INTRODUCTION TO CONFORMAL FIELD THEORY WITH GAUGE SYMMETRIES

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Introduction

Conformal field theory is a two-dimensional quantum field theory defined over compact Riemann surfaces. It was initiated by Belavin, Polyakov and Zamolodchikov, [4]. The theory is invariant under conformal transformations. Since the group of conformal transformations is of infinite dimension, conformal field theory is reduced to finite degrees of freedom by the conformal invariance. It is well-known that an oriented surface S with a Riemannian metric determines a complex structure. Moreover, the complex structure is uniquely determined by the conformal class of the metric.

Let z be a local holomorphic coordinate of the surface S. In this coordinate a conformal transformation is nothing but a change of holomorphic coordinate

$$z \mapsto w = f(z).$$

The infinitesimal version of a conformal transformation is

$$z \mapsto z + \epsilon g(z), \tag{1}$$

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where ϵ is the dual number, that is, ϵ is the class $x \pmod{x^2}$ in $\mathbb{C}[x]/(x^2)$ and $g(z)$ is holomorphic in z. Precisely speaking, it is better to regard $g(z)$ as a local holomorphic vector field $g(z)\,d/dz$. Hence, if we choose a different holomorphic local coordinate $w = f(z)$ then the infinitesimal conformal transformation corresponding to (1) is

$$w \mapsto w + \epsilon g(h(w))h'(w),$$

where $z = h(w)$. Strangely enough, in the physics literature it is written that $h(z)$ is meromorphic at the origin $z = 0$. At first glance this seems incorrect. But the physicist has good reasons to assume that $h(z)$ is meromorphic at the origin. For each integer n, put

$$\ell_n := -z^{n+1}\frac{d}{dz}.$$

Then, $\{\,\ell_n\,\}$ forms a Lie algebra under the usual bracket operation of vector fields

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}.$$

A central extension of the Lie algebra $\{\,\ell_n\,\}$ is called the *Virasoro* algebra. In conformal field theory the symmetry of the theory is given by the Virasoro algebra. This is the reason why the physicist always assumes that $h(z)$ is meromorphic.

But how can we interpret an infinitesimal conformal transformation (1) when $h(z)$ has a pole at the origin? Let U be a coordinate neighbourhood with local coordinate z and V one with local coordinate w such that $U \cap V$ does not contain the origin $z = 0$. Then, we may regard (1) as an infinitesimal change of the coordinates on $U \cap V$. This means that if $h(z)$ has a pole at the origin, then (1) defines the first order infinitesimal change of complex structure of our surface S. (For the details, see §1.2 below.) Thus conformal field theory over a Riemann surface S naturally relates to its infinitesimal deformations of the complex structure of S. This is the reason why conformal field theory relates to the moduli of Riemann surfaces. Thus, conformal field theory has much richer structures than a theory invariant under the usual conformal transformations (i.e. $h(z)$ is holomorphic).

The basic assumption of (rational) conformal field theory is that the theory can be split into the holomorphic part and the anti holomorphic part. Holomorphic conformal field theory is usually called a chiral conformal field theory. Once we can construct holomorphic conformal field theories, then by taking the complex conjugate we have anti-holomorphic conformal field theories. The guiding principle of constructing physical conformal field theories by combining holomorphic and anti-holomorphic conformal field theories is the so called modular invariance. The mathematically rigorous treatment of Conformal field theory over $\mathbb{P}^1$ was first givne by Tsuchiya and Kanie [36] and the theory was generalized over any compact Riemann surfaces by Tsuchiya, Ueno and Yamada [38]. There is another approach to conformal field theory by Segal [33].

The present notes are based on the lectures I have given at several universities. In these lectures I have tried to clarify the discussions given in [38], since [38] contains many mistakes and misprints. In the introduction of my previous notes [39], I explained how to modify the arguments of [38] to give correct proofs of the main statements of [38]. In the present notes, I give the details.

The present notes also contains several elementary results with proofs. My original idea was to write introductory lecture notes to conformal field theory. However, I will assume many basic important results on algebraic geometry, since the reader can easily find that else where. I hope the reader will get a rough idea of what conformal field theory is and how physical theories like [4], [10] and [13] can be mathematically formulated.

On the other hand, I have to omit many important recent results on conformal field theory such as the Verlinde formula (see for example, [9], [31], [32], [40]), geometric interpretation of conformal blocks and the connections (see, for example [2], [28], [30], [18]), applications to

three-dimensional topology (see for example [27], [41]), and representation of conformal blocks by means of integrals (see, for example [5], [12], [29]). Neither do I discuss abelian conformal field theory, which has interesting relationships to other branches of mathematics (see, for example [7], [23], [20], [21], [22]).

I would like to thank the organizer of the Geometry and Physics conference in Aarhus for giving me the opportunity to give these lectures on conformal field theory, the editors for inviting me to publish the present notes in the Proceedings, and the referee who pointed out several mistakes in my original notes. The final part of the proof of theorem 3.3.1 is also due to him. Last but not least I would like to express my thanks to Jørgen Ellegaard Andersen and Jakob Grove for suggesting several improvement of the present notes.

1. Compact Riemann Surfaces and Stable Curves

1.1. Pointed Stable Curves.

Definition 1.1.1. Data $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N)$ consisting of a curve C and points $Q_1, \ldots, Q_N$ on C are called an *N-pointed stable curve*, if the following conditions are satisfied.

(1) The curve C is a reduced connected complete algebraic curve defined over the complex numbers $\mathbb{C}$. The singularities of the curve C are at worst ordinary double points. That is, C is a semi-stable curve.
(2) $Q_1, Q_2, \ldots, Q_N$ are non-singular points of the curve C.
(3) If an irreducible component C_i is the projective line (i.e. the Riemann sphere) $\mathbb{P}^1$ (respectively, a rational curve with one double point, respectively, an elliptic curve), the sum of the number of intersection points of C_i with other components and the number of Q_j's on C_i is at least three (respectively, one).
(4) $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$.

Note that the above condition (3) is equivalent to saying that $\mathrm{Aut}(\mathfrak{X})$ is a finite group so that $\mathfrak{X}$ has no infinitesimal automorphisms. In the following we often add the following condition (Q) for an N-pointed stable curve $\mathfrak{X}$.

$$\text{Each component } C_i \text{ contains at least one } Q_j. \tag{Q}$$

The meaning of condition (Q) will be clarified in the following Lemmas 1.1.6 and 1.1.7.

Definition 1.1.2. Let C be a curve and Q a non-singular point on C. An *n-th infinitesimal neighbourhood* $s^{(n)}$ of C at the point Q is a $\mathbb{C}$-algebra isomorphism

$$s^{(n)} \colon \mathcal{O}_{C,Q}/\mathfrak{m}_Q^{n+1} \simeq \mathbb{C}\big[[\xi]\big]/(\xi^{n+1}), \tag{1.1.1}$$

where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{C,Q}$ consisting of germs of holomorphic functions vanishing at Q.

Taking the limit $n \to \infty$ in the isomorphism (1.1.1), we have an isomorphism

$$s^{(\infty)} \colon \widehat{\mathcal{O}}_{C,Q} \simeq \mathbb{C}\big[[\xi]\big]. \tag{1.1.2}$$

The isomorphism $s^{(\infty)}$ is called a *formal neighbourhood* of C at Q.

Definition 1.1.3. The data $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N; s_1, s_2, \ldots, s_N)$ is called an *N-pointed stable curve of genus g with formal neighbourhoods*, if

(1) $(C; Q_1, Q_2, \ldots, Q_N)$ is an N-pointed stable curve of genus g.
(2) s_j is a formal neighbourhood of C at Q_j.

Similarly one defines an N-pointed stable curve with n-th infinitesimal neighbourhoods $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)})$.

Let C be a semi-stable curve, that is, C is a reduced curve with at most ordinary double points and proper over $\mathbb{C}$. Let Ω^1_C be the sheaf of Kähler differentials of the curve C and ω_C be the dualizing sheaf of the curve C. Near a singular point P, the curve C is analytically isomorphic to the variety defined by

$$xy = 0.$$

By these coordinates the sheaf Ω^1_C is expressed as

$$\Omega^1_C = (\mathcal{O}_C dx + \mathcal{O}_C dy)/(x dy + y dx)\mathcal{O}_C. \tag{1.1.3}$$

On the other hand, near the singular point P the dualizing sheaf ω_C is an invertible sheaf generated by the differential ζ given by dx/x outside $x = 0$ and $-dy/y$ outside $y = 0$. Moreover, outside the singular points of the curve C, the sheaves Ω^1_C and ω_C coincide. Thus, we have

$$\Omega^1_C = \mathfrak{m}\omega_C, \tag{1.1.4}$$

where $\mathfrak{m}$ is the defining ideal sheaf of the singular points of C. Hence, we have the following exact sequence

$$0 \longrightarrow \Omega^1_C \longrightarrow \omega_C \longrightarrow \omega_C \otimes \mathcal{O}_{C_{\mathrm{Sing}}} \longrightarrow 0.$$

Let $\nu\colon \widetilde{C} \to C$ be the normalization of the curve C. We let $\{P_1, \ldots, P_q\}$ be the set of double points of the curve C and for each double point P_i, put $\nu^{-1}(P_i) = \{ P_{i,+}, P_{i,-} \}$. Then, we have the following exact sequence

$$0 \longrightarrow \omega_C \longrightarrow \nu_*\omega_{\widetilde{C}}\Big(\sum_i^k (P_{i,+} + P_{i,-})\Big) \xrightarrow{r} \bigoplus_i^q \mathbb{C} \longrightarrow 0, \tag{1.1.5}$$

where at each double point P_i, the mapping r is given by

$$\mathrm{res}_{P_{i,+}}(\tau) - \mathrm{res}_{P_{i,-}}(\tau).$$

This means that a local holomorphic section of the dualizing sheaf ω_C is regarded as a local meromorphic one-form on $\widetilde{C}$ which has a pole of order one at $P_{i,+}$ and $P_{i,-}$ such that the sum of the residues is zero and which is holomorphic outside $P_{i,\pm}$'s. In the following we shall often use this interpretation. The following lemma is an easy consequence of this interpretation.

Lemma 1.1.4. *Let τ be a meromorphic section of the dualizing sheaf ω_C, holomorphic at the double points. Then the sum of the residues of τ is zero.*

Lemma 1.1.5. *Assume that an N-pointed stable curve $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N; s_1, s_2, \ldots, s_N)$ with formal neighbourhoods satisfies the condition (Q). By t_j we denote the Laurent expansions at Q_j with respect to a formal parameter $\xi_j = s^{-1}(\xi)$. Then, the following homomorphisms are injective*

$$t = \oplus t_j\colon H^0\Big(C, \mathcal{O}\Big(*\sum_{j=1}^N Q_j\Big)\Big) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big), \tag{1.1.6}$$

$$t = \oplus t_j\colon H^0\Big(C, \omega_C\Big(*\sum_{j=1}^N Q_j\Big)\Big) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)\, d\xi_j, \tag{1.1.7}$$

where ω_C is the dualizing sheaf of the curve C.

Here we define

$$H^0\Big(C, \mathcal{O}\Big(*\sum_{j=1}^N Q_j\Big)\Big) = \varinjlim_{n\to\infty} H^0\Big(C, \mathcal{O}\Big(n\sum_{j=1}^N Q_j\Big)\Big),$$

where $\mathcal{O}(n\sum_{j=1}^N Q_j)$ is the sheaf of germs of meromorphic functions on C which have poles of order at most n at Q_j, $j = 1, 2, \dots, N$. $H^0\big(C, \omega_C(*\sum_{j=1}^N Q_j)\big)$ is defined similarly.

By this Lemma $H^0\big(C, \mathcal{O}(*\sum_{j=1}^N Q_j)\big)$ (respectively, $H^0\big(C, \omega_C(*\sum_{j=1}^N Q_j)\big)$) can be regarded as a subspace of $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)$ (respectively, $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$). There is the residue pairing

$$\begin{aligned} &\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big) \times \bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j \longrightarrow \mathbb{C} \\ &(f(\xi_1), \dots, f(\xi_N), g(\xi_1)d\xi_1, \dots, g(\xi_N)d\xi_N) \longmapsto \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(f(\xi_j)g(\xi_j)d\xi_j). \end{aligned} \tag{1.1.8}$$

The following Lemma is well-known and plays an important role in our theory.

Lemma 1.1.6. *Under the residue pairing the vector space*

$$H^0\Big(C, \mathcal{O}\Big(*\sum_{j=1}^N Q_j\Big)\Big)$$

and the vector space

$$H^0\Big(C, \omega_C\Big(*\sum_{j=1}^N Q_j\Big)\Big)$$

are the annihilators to each other.

Proof. On $\mathbb{C}\big((\xi_j)\big)$ and $\mathbb{C}\big((\xi_j)\big)d\xi_j$ we introduce the ξ_j-adic topology. That is, the set of open neighbourhoods of $f(\xi_j) \in \mathbb{C}\big((\xi_j)\big)$ (respectively, $f(\xi_j)d\xi_j \in \mathbb{C}\big((\xi_j)\big)d\xi_j$) consists of the set

$$f(\xi_j) + \mathbb{C}\big[[\xi_j]\big]\xi_j^k, \qquad (\text{resp.} \quad f(\xi_j)d\xi_j + \mathbb{C}\big[[\xi_j]\big]\xi_j^k d\xi_j), \quad k \in \mathbb{Z}.$$

On $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)$ and $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$ we introduce the product topology. Let $\mathcal{M}$ (respectively, $\mathcal{M}_\omega$) be the meromorphic function field (respectively, the set of meromorphic sections of the dualizing sheaf ω_C) of the curve C. For any element $(f_1(\xi_1), \dots, f_N(\xi_N)) \in \bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)$ and any positive integer m, by the Riemann-Roch theorem there is a meromorphic function f on the curve C such that

$$t_j(f) \equiv f_j(\xi_j) \pmod{\xi_j^m}.$$

Therefore, $\mathcal{M}$ is a dense subspace of $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)$. Also by the Riemann-Roch theorem $\mathcal{M}_\omega$ is a dense subspace of $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$. Assume that $(g(\xi_1)d\xi_1, \dots, g(\xi_N)d\xi_N) \in \bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$ is annihilated by $H^0\big(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j)\big)$. Assume further that the element $(g(\xi_1)d\xi_1, \dots, g(\xi_N)d\xi_N)$ comes from an element $\tau \in \mathcal{M}_\omega$: $(g(\xi_1)d\xi_1, \dots, g(\xi_N)d\xi_N) = (t_1(\tau), \dots, t_N(\tau))$. Assume that τ has a pole of order k at a point P which is different from the Q_j's. Now there exists a meromorphic function h on C which has a zero of order $m-1$ at the point P and poles only at Q_j's. Then $h\tau$ is also in $\mathcal{M}_\omega$ and by Lemma 1.1.4 we have

$$\sum \operatorname*{Res}_{Q_j}(h\tau) + \operatorname*{Res}_P(h\tau) = 0.$$

By using the coordinates ξ_j, this can be expressed in the form

$$\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(t_j(h)g_j d\xi_j) + \operatorname*{Res}_P(h\tau) = 0.$$

By our assumption the first term of the above equality is zero. Therefore, we have $\operatorname{Res}_P(h\tau) = 0$. But $h\tau$ has a pole of order one at P, hence has non-zero residue. This is a contradiction. Thus, τ is in $H^0\big(C, \omega_C(*\sum_{j=1}^N Q_j)\big)$. Since $\mathcal{M}_\omega$ is dense in $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$, this shows that $H^0\big(C, \omega_C(*\sum_{j=1}^N Q_j)\big)$ is the annihilator of $H^0\big(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j)\big)$ in $\bigoplus_{j=1}^N \mathbb{C}\big((\xi_j)\big)d\xi_j$. The other statement can be proved similarly. □

1.2. Deformations of Riemann Surfaces and Stable Curves.

Let C be a compact Riemann surface of genus g. A *deformation* of C is by definition, a proper *smooth* holomorphic mapping $\pi\colon X \to Y$ of complex spaces with a prescribed point $y \in Y$ such that $\pi^{-1}(y)$ is isomorphic to the Riemann surface C. If $Y = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$, the deformation of C is called an infinitesimal (or first order) deformation of C. The infinitesimal deformations of the Riemann surface C are parameterized by the cohomology group $H^1(C, \Theta_C)$, where Θ_C is the sheaf of germs of holomorphic vector fields on C. (See, for example [26].) More generally, we can define a deformation of the data $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_N^{(n)})$ of an N-pointed Riemann surface of genus g with n-th infinitesimal neighbourhoods. The infinitesimal deformations of an N-pointed Riemann surface of genus g with n-th infinitesimal neighbourhoods are parameterized by the cohomology group $H^1\bigl(C, \Theta_C\bigl(-(n+1)\sum_{j=1}^N Q_j\bigr)\bigr)$. If C is a singular semi-stable curve, a deformation of C is defined as a proper *flat* holomorphic mapping $\pi\colon X \to Y$ of complex spaces with a prescribed point $y \in Y$ such that $\pi^{-1}(y)$ is isomorphic to the curve C. In this case, the infinitesimal deformations of the curve C are parameterized by the cohomology group $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$ and the infinitesimal deformations of an N-pointed stable curve $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_N^{(n)})$ with n-th infinitesimal neighbourhoods are parameterized by the cohomology group $\operatorname{Ext}^1_{\mathcal{O}_C}\bigl(\Omega^1_C, \mathcal{O}_C\bigl(-(n+1)\sum_{j=1}^N Q_j\bigr)\bigr)$. (See, for example, [8, Section 1].) Here, Ω^1_C is the sheaf of Kähler differentials on the curve C. (See, (1.1.3)). In our situation, we may regard the exact sequence (1.2.3) below as a definition of the sheaf Ω^1_C. Put $\Theta_C = \underline{Hom}_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. There is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1\Bigl(C, \Theta_C\Bigl(-(n+1)\sum_{j=1}^N Q_j\Bigr)\Bigr) &\longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}\Bigl(\Omega^1_C, \mathcal{O}_C\Bigl(-(n+1)\sum_{j=1}^N Q_j\Bigr)\Bigr) \\ &\longrightarrow H^0\left(C, \underline{Ext}^1_{\mathcal{O}_C}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \longrightarrow 0. \end{aligned} \tag{1.2.1}$$

Note that the support of the sheaf $\underline{Ext}^1_{\mathcal{O}_C}\bigl(\Omega^1_C, \mathcal{O}_C\bigl(-(n+1)\sum_{j=1}^N Q_j\bigr)\bigr)$ is the singular locus of the curve C, hence the sheaf is isomorphic to $\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. If the stable curve C has q double points $P_1, P_2, \ldots, P_q$, then we have

$$\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)_Q = \begin{cases} \mathbb{C}, & \text{if } Q = P_j, \quad i = 1, 2, \ldots, q, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$H^0\left(C, \underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)\right) \simeq \mathbb{C}^q.$$

Each element of $H^1\bigl(C, \Theta_C\bigl(-(n+1)\sum_{j=1}^N Q_j\bigr)\bigr)$ corresponds to an infinitesimal deformation of the data $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_N^{(n)})$ *preserving the singularities*.

Definition 1.2.1. The data $(\pi\colon Y \to B; s_1, s_2, \ldots, s_N; \eta_1, \eta_2, \ldots, \eta_N)$ is called a (*holomorphic*) *family of N-pointed stable curves of genus g with formal neighbourhoods*, if the following conditions are satisfied:

(1) Y and B are connected complex manifolds, $\pi\colon Y \to B$ is a proper flat holomorphic map and $s_1, s_2, \ldots, s_N$ are holomorphic sections of π.
(2) For each point $b \in B$ the data $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \ldots, s_N(b))$ is an N-pointed stable curve of genus g.
(3) η_j is an $\mathcal{O}_B$-algebra isomorphism

$$\eta_j\colon \widehat{\mathcal{O}}_{/s_j} = \varprojlim_{n\to\infty} \mathcal{O}_Y/I_j^n \simeq \mathcal{O}_B\bigl[[\xi]\bigr],$$

where I_j is the defining ideal of $s_j(B)$ in Y.

Similarly we define a family $(\pi\colon Y \to B; s_1, s_2, \ldots, s_N; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \ldots, \tilde{\eta}_N^{(n)})$ of *N-pointed stable curve of genus g with n-th formal neighbourhoods*, by changing the third condition to

(3') $\tilde{\eta}_j^{(n)}$ is an $\mathcal{O}_B$-algebra isomorphism

$$\tilde{\eta}_j^{(n)}\colon \mathcal{O}_Y/I_j^{n+1} \simeq \mathcal{O}_B\left[[\xi]\right]/I_j^{(n+1)}.$$

For a holomorphic mapping $f\colon D \to B$, we can define the pullback $f^*\mathfrak{X}^{(n)}$ (respectively, $f^*\mathfrak{X}$) of the family of N-pointed stable curves with n-th infinitesimal neighbourhoods (respectively, formal neighbourhoods) over the base space D by the mapping f.

Definition 1.2.2. A family $\mathfrak{X} = (\pi^{(n)}\colon \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \ldots, \tilde{\eta}_N^{(n)})$ of N-pointed stable curves of genus g with n-th formal neighbourhoods is called *versal* (respectively, *universal*) at a point $b \in \mathcal{B}^{(n)}$, if for any deformation $\mathfrak{Y} = (\pi\colon X \to Y; s_1, \ldots, s_N; \widehat{\eta}_1, \ldots, \widehat{\eta}_N)$ of $\pi^{(n)-1}(b) = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ with prescribed point $y \in Y$ there exists a holomorphic mapping (respectively, unique holomorphic mapping) f from a neighbourhood of y in Y to $\mathcal{B}^{(n)}$ such that the pullback $f^*\mathfrak{X}$ is isomorphic to $\mathfrak{Y}$ in a neighbourhood of y in Y and such that df is uniquely determined at the prescribed point y. If the family is versal (respectively, universal) at each point of $\mathcal{B}^{(n)}$, the family $\mathfrak{X}$ is called a versal (respectively, universal) family.

In the following we mainly consider a family of pointed stable curves with formal coordinates, which is versal as a family of pointed stable curves. Hence, in this section we only consider a family of pointed stable curves.

Example 1.2.3. Let H be the upper half plane. We let $\mathbb{Z}^2$ act on $H \times \mathbb{C}$ by

$$\begin{aligned}(m, n)\colon H \times \mathbb{C} &\longrightarrow H \times \mathbb{C},\\ (\tau, \zeta) &\longmapsto (\tau, \zeta + m\tau + n).\end{aligned}$$

Then $\mathbb{Z}^2$ acts on $H \times \mathbb{C}$ properly discontinuously and freely, hence the quotient space $\mathcal{E} = (H \times \mathbb{C})/\mathbb{Z}^2$ is a complex manifold. We let $(\tau, [\zeta])$ be a point of $\mathcal{E}$ corresponding to a point (τ, ζ) of $H \times \mathbb{C}$. Then the natural mapping $p\colon \mathcal{E} \to H$ defined as

$$p\colon \big(\tau[\zeta]\big) \longmapsto \tau$$

induced by the projection onto the first factor of $H \times \mathbb{C}$ is a proper smooth holomorphic mapping whose fiber at a point $\tau \in H$ is an elliptic curve with period matrix $(1, \tau)$. The family $p\colon \mathcal{E} \to H$ is a versal family.

Theorem 1.2.4. *For a family $(\pi\colon Y \to B; s_1, s_2, \ldots, s_N)$ of N-pointed stable curve of genus g and for each point $b \in B$, there exists a $\mathbb{C}$-linear mapping*

$$\rho_b : T_bB \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{Y_b}}\Big(\Omega^1_{Y_b}, \mathcal{O}_{Y_b}\Big(-\sum_{j=1}^{N} s_j(b)\Big)\Big), \tag{1.2.2}$$

where $Y_b = \pi^{-1}(b)$.

The $\mathbb{C}$-linear mapping ρ_b is called the *Kodaira-Spencer mapping* of the family $(\pi\colon Y \to B; s_1, s_2, \ldots, s_N;)$ at the point b.

Since the theorem plays an important role in our formulation of conformal field theory, we give a rather detailed discussions of the proof. For the fundamental properties of the functor Ext we refer the reader to [17, Chap. 3, 6].

Put $C = Y_b$, $Q_j = s_j(b)$. Let I_C be the sheaf of the defining ideal of C in Y. There is an exact sequence

$$0 \longrightarrow I_C/I_C^2 \longrightarrow \Omega^1_Y \otimes \mathcal{O}_C \longrightarrow \Omega^1_C \longrightarrow 0. \tag{1.2.3}$$

This gives a locally free resolution of the sheaf Ω^1_C. The sheaf I_C/I_C^2 is the conormal sheaf of the curve C in Y and we have a canonical isomorphism

$$(T_b^* B) \otimes_{\mathbb{C}} \mathcal{O}_C \simeq I_C/I_C^2.$$

Hence there are canonical isomorphisms

$$\underline{Hom}_{\mathcal{O}_C}(I_C/I_C^2, \mathcal{O}_C) \simeq T_b B \otimes_{\mathbb{C}} \mathcal{O}_C, \tag{1.2.4}$$

$$\underline{Hom}_{\mathcal{O}_C}(\Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \mathcal{O}_C) \simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C. \tag{1.2.5}$$

Put

$$I^0 = \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \quad I^1 = T_b B \otimes_{\mathcal{O}_C} \mathcal{O}_C.$$

Then, we have an exact sequence

$$0 \longrightarrow \Theta_C \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \longrightarrow 0.$$

Applying $\underline{Hom}_{\mathcal{O}_C}(\cdot, \mathcal{O}_C)$ to the exact sequence (1.2.3) and using the canonical isomorphisms (1.2.4) and (1.2.5), we obtain a complex of sheaves

$$0 \longrightarrow I^0 \xrightarrow{\pi_*} I^1 \longrightarrow 0. \tag{1.2.6}$$

The cohomology groups of the complex (1.2.6) are $\underline{Ext}^\bullet_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. That is, we have

$$\underline{Ext}^0_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) = \mathrm{Ker}\{\pi_* : I^0 \to I^1\} = \Theta_C,$$
$$\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) = \mathrm{coker}\{\pi_* : I^0 \to I^1\}.$$

Note that the map π_* in (1.2.6) is surjective outside the double points $P_1, P_2, \ldots, P_q$ of the curve C. The cohomology groups $\mathrm{Ext}^\bullet_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is calculated as follows. Choose an open covering $\mathcal{U} = \{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of the curve C. Let $C^k(\mathcal{U}, I^m)$ be k-th cochains with values in the sheaf I^m. Put

$$K^n = \bigoplus_{k+m=n} C^k(\mathcal{U}, I^m).$$

We define the differentials δ^n of $\{K^n\}$ as follows. For any element $\{\phi_\alpha\} \in C^0(\mathcal{U}, I^0) = K^0$ we define

$$\delta^0\{\phi_\alpha\} = \big(\{\pi_*(\phi_\alpha)\}, \{\phi_\beta - \phi_\alpha\}\big) \in C^0(\mathcal{U}, I^1) \oplus C^1(\mathcal{U}, I^0) = K^1.$$

For each element $(\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\}) \in C^0(\mathcal{U}, I^1) \oplus C^1(\mathcal{U}, I^0) = K^1$ we define

$$\delta^1(\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\}) = \big(\{(\varphi_\beta - \varphi_\alpha) - \pi_*(\theta_{\alpha\beta})\}, \{\theta_{\beta\gamma} - \theta_{\alpha\gamma} + \theta_{\alpha\beta}\}\big)$$

as an element in $C^1(\mathcal{U}, I^1) \oplus C^2(\mathcal{U}, I^0) = K^2$. The other δ^k's are defined to be the zero map. Then $\{K^\bullet, \delta^\bullet\}$ is a complex and if the covering is good, namely each open set $\mathcal{U}_\lambda$ is different from C, then we have

$$\mathrm{Ext}^n_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) = \mathrm{Ker}\,\delta^n / \mathrm{Im}\,\delta^{n-1}.$$

Assume that the covering $\mathcal{U}$ is good. Assume further that each of the points Q_j's and P_i's is contained in only one open set $\mathcal{U}_\gamma$. For each tangent vector $\theta \in T_b B$ of B at b, there is a vector field $\tilde{\theta}$ on a neighbourhood of b. Then there is a lifting $\tilde{\theta}_\alpha$ on $\widetilde{\mathcal{U}}_\alpha \setminus \Sigma$ of the vector field $\tilde{\theta}$, where $\widetilde{\mathcal{U}}_\alpha$ is an open set in Y with $\mathcal{U}_\alpha = \widetilde{\mathcal{U}}_\alpha \cap C$ and Σ is the locus of double points of the fibers of π. Put

$$\varphi_\alpha = \theta \in H^0(\mathcal{U}_\alpha, T_b B \otimes_{\mathcal{O}_C} \mathcal{O}_C),$$
$$\theta_{\alpha\beta} = (\tilde{\theta}_\beta - \tilde{\theta}_\alpha)|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \in H^0(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C).$$

Then $\Psi(\theta) = (\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\})$ is an element of K^1 and by definition we have $\delta^1(\Psi(\theta)) = 0$, hence it defines an element $[\Psi(\theta)]$ of $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. Thus we have a $\mathbb{C}$-linear mapping

$$\rho_b : T_b B \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C).$$

This is the Kodaira-Spencer mapping of $\pi : Y \to B$ at b.

So far we have not considered the points $Q_j = s_j(b)$. To define the Kodaira-Spencer mapping of the family $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$ we need to be careful about how we choose a lifting $\tilde{\theta}_\alpha$ of $\tilde{\theta}$, namely the lifting should respects the points $Q_j = s_j(b)$. For simplicity assume that the point Q_j is contained in only one open set $\mathcal{U}_j$. Choose local coordinates $(u_1, u_2, \ldots, u_m)$ of B with center b. Then we can choose local coordinates of Y with center Q_j as $(u_1, u_2, \ldots, u_m, z)$. We may assume that $\mathcal{U}_j$ is contained in the coordinate neighbourhood of Q_j with the above coordinates. By these coordinates the vector field $\tilde{\theta}$ is expressed in the form $\sum a_k(u)\frac{\partial}{\partial u_k}$. Then, for $\tilde{\theta}_j$ we choose the same form $\sum a_k(u)\frac{\partial}{\partial u_k}$. Any other lifting is of the form

$$\sum a_k(u)\frac{\partial}{\partial u_k} + A(u,z)\frac{\partial}{\partial z}.$$

To preserve the points Q_j, $A(u,z)$ has to have a zero of order 1 at Q_j. Precisely speaking, if we choose the lifting $\tilde{\theta}_j$ above, then we have an element $\Psi(\theta)$ as above. This lifting *does* depend on the choice of local coordinates. If we choose other local coordinates, $\Psi(\theta)$ changes by thr addition of $\delta^0(\{\phi_\alpha\})$. Now ϕ_α corresponds to an infinitesimal change of local coordinates of $\mathcal{U}_\alpha$. Hence ϕ_α needs to preserve the points Q_j. Let I_1^0 and I_1^1 be $\mathcal{O}_C$-submodules of I^0 and I^1, respectively defined by

$$I_1^0 = \underline{Hom}_{\mathcal{O}_C}\Big(\Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \mathcal{O}_C\Big(-\sum_{j=1}^n Q_j\Big)\Big) \simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C\Big(-\sum_{j=1}^N Q_j\Big),$$

$$I_1^1 = \underline{Hom}_{\mathcal{O}_C}\Big(I_C/I_C^2, \mathcal{O}_C\Big(-\sum_{j=1}^N Q_j\Big)\Big) \simeq T_b B \otimes_{\mathbb{C}} \mathcal{O}_C\Big(-\sum_{j=1}^N Q_j\Big).$$

Then, we also have a complex of sheaves

$$0 \longrightarrow I_1^0 \xrightarrow{\pi_*} I_1^1 \longrightarrow 0$$

and $(\{\phi_\alpha\}, \{\theta_{\alpha\beta}\})$ defines a cohomology class $[\Psi(\theta)]$ of the complex $\{K_1^\bullet, \delta^\bullet\}$, where we define

$$K_1^p = \bigoplus_{m+\ell=p} C^m(\mathcal{U}, I_1^\ell).$$

The cohomology groups of $\{K_1^\bullet, \delta^\bullet\}$ are $\mathrm{Ext}^\bullet_{\mathcal{O}_C}\big(\Omega^1_C, \mathcal{O}_C(-\sum Q_j)\big)$. Hence $[\Psi(\theta)]$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_C}\big(\Omega^1_C, \mathcal{O}_C\big(-\sum Q_j\big)\big)$. This defines the Kodaira-Spencer mapping of the family $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$. We have thus proved Theorem 1.2.4.

A sheaf version of Theorem 1.2.4 is the following:

Corollary 1.2.5. *If $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$ is a family of N-pointed* smooth *curves of genus g, the Kodaira-Spencer mapping ρ_b induces an $\mathcal{O}_B$-module homomorphism*

$$\rho : \Theta_B \longrightarrow R^1\pi_* \underline{Hom}\Big(\Omega^1_{Y/B}, \mathcal{O}_Y\Big(-\sum_{j=1}^N s_j(B)\Big)\Big).$$

Proposition 1.2.6. *A family $(\pi\colon \mathcal{C} \to \mathcal{B}; s_1, s_2, \ldots, s_N)$ of N-pointed stable curves of genus g is versal at a point $b \in \mathcal{B}$, if and only if the Kodaira-Spencer mapping*

$$\rho_b\colon T_bB \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C_b}}\left(\Omega^1_{\mathcal{O}_{C_b}}, \mathcal{O}_{C_b}\left(-\sum_{j=1}^{N} s_j(b)\right)\right)$$

is an isomorphism at the point b.

Remark 1.2.7. For a smooth family $\pi\colon X \to Y$ of compact complex manifolds over a complex manifold Y, the Kodaira-Spencer mapping has the following description. Let U be an open neighbourhood of a point b of Y and $\{\mathcal{U}_i\}_{i\in I}$ is a small open covering of $\pi^{-1}(U)$. Let $(z_1, \ldots, z_m)$ be local coordinates of U with center b and $(u_i^1, \ldots, u_i^n, z_1, \ldots, z_m)$ be local coordinates of $\mathcal{U}_i$. On $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$ we have

$$u_i^k = h_{ij}^k(u_j^1, \ldots, u_j^n, z_1, \ldots, z_m).$$

Then, for a vector field v on U, $\rho_b(v)$ is given by

$$\sum_{k=1}^{n} v(h_{ij}^k)(u_j^1, \ldots, u_j^n, 0 \ldots, 0)\frac{\partial}{\partial u_i^k}.$$

In this way we have the Kodaira-Spencer mapping $\rho_b\colon T_bY \to H^1(X_b, \Theta_{X_b})$, where we put $X_b = \pi^{-1}(b)$.

Let us calculate the Kodaira-Spencer class of the family of elliptic curves constructed in Example 1.2.3.

Example 1.2.8. Let us first consider an elliptic curve $E = \mathbb{C}/(1, \tau_0)$ $\tau_0 \in H$. Let z be a global coordinate of the affine space $\mathbb{C}$ and o the origin of the elliptic curve E. From the exact sequence

$$0 \longrightarrow \Theta_E \longrightarrow \Theta_E(*o) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}]\frac{d}{dz} \longrightarrow 0,$$

we have an exact sequence

$$0 \longrightarrow H^0(E, \Theta_E(*o))/H^0(E, \Theta_E) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}]\frac{d}{dz} \xrightarrow{\theta} H^1(E, \Theta_E) \longrightarrow 0,$$

where we put

$$\Theta_E(*o) = \varinjlim_{m\to\infty} \Theta(mo),$$
$$H^0(E, \Theta_E(*o)) = \varinjlim_{m\to\infty} H^0(E, \Theta(mo)).$$

Note that for the Weierstrass $\wp$-function, we have

$$\frac{(-1)^n}{n!}\wp^{(n)}(z)\frac{d}{dz} \in H^0\left(E, \mathcal{O}_E\left((n+2)o\right)\right) \setminus H^0\left(E, \mathcal{O}_E((n+1)o)\right)$$

for all non-negative integers n. Therefor, $z^{-1}\frac{d}{dz}$ has a non-zero image $\theta(z^{-1}\frac{d}{dz})$ in $H^1(E, \Theta_E)$. Let us describe the image. Let $\{\mathcal{U}_\lambda\}_{\lambda=0}^N$ be a small open covering of E such that $\mathcal{U}_0$ is a coordinate neighbourhood of o with local coordinate z and such that

$$o \notin \mathcal{U}_0 \cap \mathcal{U}_\mu, \quad \mu \geqslant 1.$$

Then the image of $z^{-1}\frac{d}{dz}$ is given by a cocycle

$$\tau_{\lambda\mu} = \begin{cases} z^{-1}\frac{d}{dz}, & \lambda = 0, \quad \mu \geqslant 1, \quad \mathcal{U}_{\lambda\mu} \neq \varnothing, \\ -z^{-1}\frac{d}{dz}, & \lambda \geqslant 1, \quad \mu = 0, \quad \mathcal{U}_{\lambda\mu} \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Let $p\colon \mathcal{E} \to H$ be the versal family constructed in Example 1.2.3. As we put $E = \mathbb{C}/(1, \tau_0)$, by using the above covering, we find an open covering $\{\widehat{\mathcal{U}}_\lambda\}_{\lambda=0}^N$ of $p^{-1}(V)$ where V is an open neighbourhood of τ_0 such that $p^{-1}(\tau_0) \cap \widehat{\mathcal{U}}_\lambda = \mathcal{U}_\lambda$. Let $(\tau_\lambda, z_\lambda)$ be local coordinates of $\widehat{\mathcal{U}}_\lambda$ such that

$$\begin{aligned} \tau_\lambda &= \tau - \tau_0, \\ z_0 &= z, \\ z_\lambda &= z_\mu + m_{\lambda\mu} + n_{\lambda\mu}\tau, \end{aligned}$$

with $m_{\lambda\mu}$, $n_{\lambda\mu} \in \mathbb{Q}$. Then, the Kodaira-Spencer class $\rho_{\tau_0}(\frac{\partial}{\partial\tau}) \in H^1(E, \Theta_E)$ is given by the cocycle

$$\theta_{\lambda\mu} = \left\{ n_{\lambda\mu} \frac{d}{dz_\lambda} \right\}.$$

Let us consider the Dolbeault representations of the cocycles $\{\tau_{\lambda\mu}\}$ and $\{\theta_{\lambda\mu}\}$.

First consider the cocycle $\{\theta_{\lambda\mu}\}$. Put

$$\psi_\lambda = \frac{(\bar{z}_\lambda - z_\lambda)}{2\sqrt{-1}\,\mathrm{Im}\,\tau} \frac{d}{dz_\lambda}.$$

Then we have

$$\psi_\mu - \psi_\lambda = n_{\lambda\mu} \frac{d}{dz_\lambda} = \theta_{\lambda\mu}.$$

Therefore

$$\overline{\partial}\psi_\mu = \overline{\partial}\psi_\lambda = (2\sqrt{-1}\,\mathrm{Im}\,\tau)^{-1} \frac{d}{dz} d\bar{z}$$

is the corresponding Dolbeault representative of $\{\theta_{\lambda\mu}\}$.

Next consider the cohomology class $\{\tau_{\lambda\mu}\}$. We may assume that

$$\{\, z \in \mathbb{C} \mid |z| \leqslant \epsilon \,\} \subset \mathcal{U}_0$$

and

$$\{\, z \in \mathbb{C} \mid |z| \leqslant \epsilon/2 \,\} \subset \mathcal{U}_0 \setminus \bigcup_{\lambda=1}^N \mathcal{U}_\lambda.$$

Let $f(z)$ be a real valued C^∞ function in $\mathcal{U}_0$ such that $0 \leqslant f(z) \leqslant 1$ and

$$f(z) = \begin{cases} 0, & |z| < \epsilon/3 \\ 1, & |z| \geqslant \epsilon/2. \end{cases} \tag{1.2.7}$$

Put

$$\widetilde{\tau}_\mu = \begin{cases} \frac{f(z)}{z} \frac{d}{dz} & \mu = 0 \\ 0 & \mu \geqslant 1. \end{cases}$$

Then, we have that

$$\tau_{\lambda\mu} = \widetilde{\tau}_\mu - \widetilde{\tau}_\lambda$$

and

$$\omega = \overline{\partial}\widetilde{\tau}_\lambda = \frac{1}{z} \left(\frac{\partial f(z)}{\partial \bar{z}} \right) \frac{d}{dz} d\bar{z}$$

represents the Dolbeault cohomology class of $\{\tau_{\lambda\mu}\}$. Since the Dolbeault cohomology class has a representative of the form

$$a \frac{d}{dz} d\bar{z}, \quad a \in \mathbb{C},$$

let us find the constant a for ω. Note that for any C^∞ function h on E we get the equality

$$\int_E (\omega + \overline{\partial} h) \wedge dz = \int_E \omega \wedge dz.$$

We have

$$\begin{aligned}\int_{E-\{\,|z|<\epsilon\,\}} \frac{\partial f(z)}{\partial \bar{z}} \frac{1}{z} d\bar{z} \wedge dz &= \int_{E-\{\,|z|<\epsilon\,\}} \overline{\partial}\left(\frac{f(z)}{z} dz\right) \\ &= \int_{E-\{\,|z|<\epsilon\,\}} d\left(\frac{f(z)}{z} dz\right) \\ &= -\int_{|z|=\epsilon} \frac{1}{z} dz = -2\pi\sqrt{-1}\end{aligned}$$

by (1.2.7). On the other hand, we have

$$\int_E a\, d\bar{z} \wedge dz = 2\sqrt{-1} a \operatorname{Im} \tau.$$

Therefore we conclude that $\{\tau_{\lambda\mu}\}$ and $-\frac{\pi}{\operatorname{Im}\tau}\frac{d}{dz}d\bar{z}$ represent the same cohomology class. Thus we obtain the equality

$$\theta\left(z^{-1}\frac{d}{dz}\right) = -2\pi\sqrt{-1}\rho\left(\frac{\partial}{\partial\tau}\right).$$

Next let us consider another coordinate. Put

$$w = \exp(2\pi\sqrt{-1}z), \quad q_0 = \exp(2\pi\sqrt{-1}\tau_0).$$

Then the elliptic curve E can be regarded as the quotient manifold $\mathbb{C}^*/\langle q_0\rangle$ where $\langle q_0\rangle$ means the infinite cyclic group acting on $\mathbb{C}^*$ generated by an analytic automorphism

$$w \mapsto q_0 w.$$

In this coordinate, the origin of the elliptic curve corresponds to the point $w = 1$. Put

$$u = w - 1.$$

Then we have an exact sequence

$$0 \longrightarrow \Theta_E \longrightarrow \Theta_E(*o) \xrightarrow{\eta} u^{-1}\mathbb{C}[u^{-1}]\frac{d}{du} \longrightarrow 0$$

and from this exact sequence we have an exact sequence

$$0 \longrightarrow H^0(E, \Theta_E(*o))/H^0(E, \Theta_E(*o)) \xrightarrow{t} u^{-1}\mathbb{C}[u^{-1}]\frac{d}{du} \xrightarrow{\theta} H^1(E, \Theta_E) \longrightarrow 0.$$

The image $\theta(u^{-1}\frac{d}{du})$ is calculated as follows. The image is defined by a cocycle

$$\omega_{\lambda\mu} = \begin{cases} u^{-1}\frac{d}{du}, & \text{if } \lambda = 0,\ \mu \geqslant 1, \quad \mathcal{U}_{0\mu} \neq 0, \\ -u^{-1}\frac{d}{du}, & \text{if } \lambda \geqslant 1,\ \mu = 0, \quad \mathcal{U}_{\lambda 0} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since we have

$$u^{-1}\frac{d}{du} = \frac{1}{(\exp(2\pi\sqrt{-1}z)-1)2\pi\sqrt{-1}\exp(2\pi\sqrt{-1}z)}\frac{d}{dz}$$
$$= \left(\frac{-1}{4\pi^2 z} + \text{holomorphic}\right)\frac{d}{dz},$$

we conclude

$$\{\omega_{\lambda\mu}\} = -\frac{1}{4\pi^2}\{\tau_{\lambda\tau}\},$$

that is,

$$\theta\left(u^{-1}\frac{d}{dz}\right) = -\frac{1}{4\pi^2}\theta\left(z^{-1}\frac{d}{dz}\right) = -\frac{1}{2\pi\sqrt{-1}}\rho\left(\frac{d}{d\tau}\right).$$

In the following we need to consider degenerations of families of elliptic curves. In that case we shall use the parameter

$$q = \exp(2\pi\sqrt{-1}\tau).$$

Then we have

$$q\frac{d}{dq} = \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau},$$
$$\theta\left(u^{-1}\frac{d}{du}\right) = -\theta\left(q\frac{d}{dq}\right).$$

This formula will be used in section 5.5.

Now we prove the existence of the versal family of N-pointed stable curves.

Theorem 1.2.9. *For each N-pointed stable curve $(C; Q_1, \ldots, Q_N)$ of genus g, there exists a family $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \ldots, s_N)$ with prescribed point $b \in \mathcal{B}$ such that $\pi^{-1}(b)$ is isomorphic to $(C; Q_1, \ldots, Q_N)$ and such that the family $\mathfrak{F}$ is versal at b. Moreover, $\mathcal{C}$ and $\mathcal{B}$ are complex manifolds and the family $\mathfrak{F}$ is versal at each point of a small neighbourhood of b in $\mathcal{B}$. If the N-pointed stable curve has a trivial automorphism group, then the family $\mathfrak{F}$ is universal at b.*

Proof. The theorem is a consequence of deformation theory. Since we need an explicit description of a versal family, we give here a method to construct a versal family of the first infinitesimal neighbourhoods.

By our assumption, the curve C has only ordinary double points. Hence, by deformation theory, there exists a versal family $\pi^{(\varnothing)} : \mathcal{C}^{(\varnothing)} \to \mathcal{B}^{(\varnothing)}$ with specified point $x \in \mathcal{B}^{(\varnothing)}$ such that $C_x = (\pi^{(\varnothing)})^{-1}(x) \simeq C$. Hence, the Kodaira-Spencer mapping

$$\rho_x : T_x\mathcal{B}^{(\varnothing)} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C_x}}(\Omega^1_{C_x}, \mathcal{O}_{C_x})$$

is an isomorphism. (Since the automorphism group of C may not be trivial, the family $\pi : \mathcal{C}^{(\varnothing)} \to \mathcal{B}^{(\varnothing)}$ may not be universal at the point x, but only versal.) Put

$$\mathcal{B} = (\mathcal{C}^{(\varnothing)})^N \setminus \left(\bigcup_{i<j}\Delta_{ij} \bigcup \{\text{ singular points of } (\mathcal{C}^{(\varnothing)})^N\}\right),$$

where

$$\Delta_{ij} = \{(x_1, \ldots, x_N) \in \mathcal{C}^{(\varnothing)N} \mid x_i = x_j\}$$

is the (i,j)-th diagonal and $(\mathcal{C}^{(\varnothing)})^N$ means the fibred product of N copies of $\mathcal{C}^{(\varnothing)}$ over $\mathcal{B}^{(\varnothing)}$. There is a natural holomorphic mapping $p : \mathcal{B} \to \mathcal{B}^{(\varnothing)}$. Put also

$$\mathcal{C} = \mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \mathcal{B}$$

and let $\pi\colon \mathcal{C} \to \mathcal{B}$ be the projection to the second factor. By definition, $(Q_1, \dots, Q_N) \in p^{-1}(x)$. Put $x_0 = (Q_1, \dots, Q_N) \in \mathcal{B}$. Then we have $\pi^{-1}(x_0) = C_x \times x_0 \simeq C$. Moreover, we can define holomorphic sections

$$s_j\colon \mathcal{B} \longrightarrow \mathcal{C}$$

by

$$s_j((P_1, \dots P_N)) = (P_j, P_1, \dots, P_N) \in \mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \mathcal{B}.$$

Then we have $s_j(x_0) = (Q_j, x_0)$. It is easy to show that $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N)$ is versal at each point of $\mathcal{B}$. $\square$

Let $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, s_2, \dots, s_N)$ be a versal family of N-pointed stable curve of genus g. Put

$$\Sigma = \{\, P \in \mathcal{C} \mid d\pi_P\colon T_P\mathcal{C} \to T_{\pi(P)}\mathcal{B} \text{ is not surjective} \,\}, \tag{1.2.8}$$

$$D = \pi(\Sigma). \tag{1.2.9}$$

The set Σ is called the *critical locus* of the family and D is called the *discriminant locus* of the family. The following lemma is a consequence of the deformation theory of singular curves with ordinary double points. (See for example [8, Section 1].)

Lemma 1.2.10. *For a versal family of N-pointed stable curve of genus g $(\pi\colon \mathcal{C} \to \mathcal{B}; s_1, s_2, \dots, s_N)$, assume $2g - 2 + N \geqslant 1$.*

(1) *We have*

$$\dim \mathcal{B} = 3g - 3 + N$$
$$\dim \mathcal{C} = 3g - 2 + N.$$

(2) *The critical locus Σ is a smooth subvariety of codimension 2 in $\mathcal{C}$.*
(3) *The discriminant locus D is a divisor with normal crossings in $\mathcal{B}$.*

Remark 1.2.11. In the following we mainly consider versal families of N-pointed stable curves. Even if we do not consider not necessarily need to consider a versal family $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}, s_1, \dots, s_N)$, by virtue of Theorem 1.2.9, we can always assume that $\mathcal{C}$ and $\mathcal{B}$ are complex manifolds.

Example 1.2.12. Let $(\mathbb{P}^1; Q_1, \dots, Q_N)$ be an N-pointed projective line (smooth curve of genus 0). By an automorphism of $\mathbb{P}^1$, the first three points Q_1, Q_2, Q_3 can be mapped to the points 0, 1, ∞. We may regard $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$. Let u be a global coordinate on $\mathbb{C}$. Thus, $(\mathbb{P}^1; Q_1, \dots, Q_N)$ is isomorphic to $(\mathbb{P}^1; 0, 1, \infty, u_1, \dots, u_{N-3})$. Put

$$B_m = \{\, (u_1, \dots, u_m) \in \mathbb{C} \mid u_j \neq 0, 1, \infty, u_i \neq u_j, i \neq j \,\}.$$

Let $\pi\colon \mathcal{C} = \mathbb{P}^1 \times B_m \to B_m$ be the projection to the second factor. For any point $(u_1, \dots, u_m) \in B_m$ put

$$s_k((u_1, \dots, u_m)) = k, \qquad k = 0, 1, \infty,$$
$$\sigma_i((u_1, \dots, u_m)) = u_i, \qquad i = 1, 2, \dots, m.$$

Note that s_k and σ_i are holomorphic sections of π. Now, B_m is the moduli space of ordered $(m+3)$-pointed projective line and the family

$$(\pi\colon \mathcal{C} \to B_m; s_0, s_1, s_\infty; \sigma_1, \dots, \sigma_m)$$

is the universal family of ordered an $(m+3)$-pointed projective lines. The extension of the universal family to that of $(m+3)$-pointed stable curves of genus 0 was first studied by Terada [35] in a different context. The detailed study of the moduli space of N-pointed stable curves of

genus 0 was done by Terada [35] and Gerritzen, Herrlich and van der Put [16] independently. In case $m = 1$ the extension can be easily described. The compactification $\overline{B}_1$ of B_1 is $\mathbb{P}^1$. Let $\widetilde{\mathcal{C}}$ be the blowing up of $\mathbb{P}^1 \times \overline{B}_1$ at the points $(0,0)$, $(1,1)$ and (∞,∞) and $\widetilde{\pi}\colon \widetilde{\mathcal{C}} \to \overline{B}_1$ be the natural holomorphic mapping. Then, there are natural extensions $\widetilde{s}_k$ and $\widetilde{\sigma}_i$ of s_k and σ_1, respectively, which are holomorphic sections of $\widetilde{\pi}$. Then, $(\widetilde{\pi}\colon \widetilde{\mathcal{C}} \to \overline{B}_1; \widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_\infty, \widetilde{\sigma}_1)$ is the universal family of ordered four pointed stable curves of genus 0. The fibers of $\widetilde{\pi}$ over the compactified points 0, 1, ∞ are four pointed stable curves of genus 0.

By Knudsen [25], the compactification $\overline{B}_2$ of B_2 by means of ordered five pointed stable curves of genus 0 is nothing but our $\widetilde{\mathcal{C}}$. For the explicit description by means of coordinates we refer the reader to [35] and [16].

1.3. Versal Families of Pointed Stable Curves.

Let us consider a versal family $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, s_2, \ldots, s_N)$ of N-pointed stable curves of genus g. We assume that we have a local (formal) coordinate η with center $s_N(\mathcal{B})$. In the following we need to consider a family $\mathfrak{F}$ locally. For that purpose we introduce the following local coordinates on $\mathcal{C}$.

For a point $P \in \Sigma$ of the critical locus of π, we can choose local coordinates $(u_1, u_2, \ldots, u_{M-1}, z, w)$ on $\mathcal{C}$ with center P and local coordinates $(\tau_1, \tau_2, \ldots, \tau_M)$ on $\mathcal{B}$ with center $\pi(P)$ such that the holomorphic mapping π is given by

$$(u_1, u_2, \ldots, u_{M-1}, z, w) \longmapsto (u_1, u_2, \ldots, u_{M-1}, zw) = (\tau_1, \tau_2, \ldots, \tau_M).$$

In other word, we have

$$\pi^*\tau_k = \begin{cases} u_k, & k = 1, 2, \ldots, M-1 \\ zw, & k = M. \end{cases}$$

For a point $P \in \mathcal{C} \setminus \Sigma$ we can choose local coordinates $(u_1, u_2, \ldots, u_M, z)$ of $\mathcal{C}$ with center P and local coordinates $(\tau_1, \tau_2, \ldots, \tau_M)$ of $\mathcal{B}$ with center $\pi(P)$ such that the holomorphic mapping is given by

$$(u_1, u_2, \ldots, u_{M-1}, z) \longmapsto (u_1, u_2, \ldots, u_M) = (\tau_1, \tau_2, \ldots, \tau_M).$$

The $\mathcal{O}_\mathcal{C}$-module $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is defined by the following exact sequence

$$\pi^{-1}\Omega^1_\mathcal{B} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_\mathcal{C} \longrightarrow \Omega^1_\mathcal{C} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{B}} \longrightarrow 0.$$

The sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is called the *sheaf of germs of relative 1-forms* of the family $\pi\colon \mathcal{C} \to \mathcal{B}$. Let us describe the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$, by using the above local coordinates. In a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$, the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is locally isomorphic to $\mathcal{O}_\mathcal{C} dz$. In a small neighbourhood of $P \in \Sigma$, we have an $\mathcal{O}_\mathcal{C}$-module isomorphism

$$\Omega^1_{\mathcal{C}/\mathcal{C}} \simeq (\mathcal{O}_\mathcal{C} dz + \mathcal{O}_\mathcal{C} dw)/\mathcal{O}_\mathcal{C}(w dz + z dw). \tag{1.3.1}$$

Moreover, we have the following lemma.

Lemma 1.3.1. *The following sequence*

$$0 \longrightarrow \pi^{-1}\Omega^1_\mathcal{B} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_\mathcal{C} \longrightarrow \Omega^1_\mathcal{C} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{B}} \longrightarrow 0 \tag{1.3.2}$$

is exact and gives a locally free resolution of the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$.

Let $\omega_{\mathcal{C}/\mathcal{B}}$ be the relative dualizing sheaf of $\pi\colon \mathcal{C} \to \mathcal{B}$. Since $\mathcal{C}$ and $\mathcal{B}$ are non-singular and π is flat, we have an $\mathcal{O}_\mathcal{C}$-module isomorphism

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \omega_\mathcal{C} \otimes (\pi^*\omega_\mathcal{B}^{-1})$$

where ω_Y is the dualizing sheaf (canonical sheaf) of a complex manifold Y. The relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{B}}$ is described locally as follows. (See (1.1.3).)

In a small neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$, we have

$$\omega_{\mathcal{C}/\mathcal{B}} = \Omega^1_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} dz.$$

In a small neighbourhood of a point $P \in \Sigma$, we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}}(dz \wedge dw) \otimes (d\tau_M)^{-1}.$$

In particular, we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \begin{cases} \mathcal{O}_{\mathcal{C}} \frac{dz}{z}, & \text{if } z \neq 0, \\ \mathcal{O}_{\mathcal{C}} \frac{dw}{w}, & \text{if } w \neq 0, \end{cases}$$

with the relation

$$\frac{dz}{z} + \frac{dw}{w} = 0,$$

if $zw \neq 0$.

Lemma 1.3.2. *There exits an exact sequence*

$$0 \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{B}} \longrightarrow \omega_{\mathcal{C}/\mathcal{B}} \longrightarrow \omega_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_\Sigma \longrightarrow 0.$$

Proof. The mapping $\Omega^1_{\mathcal{C}/\mathcal{B}} \to \omega_{\mathcal{C}/\mathcal{B}}$ is given locally in a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$ by

$$dz \longmapsto dz$$

and in a neighbourhood of a point $P \in \Sigma$ by

$$\begin{aligned} dz &\longmapsto z(dz \wedge dw) \otimes (d\tau_M)^{-1} \\ dw &\longmapsto w(dz \wedge dw) \otimes (d\tau_M)^{-1}. \end{aligned}$$

In particular, we have

$$\begin{aligned} dz &\longmapsto z\frac{dz}{z}, \quad \text{if } z \neq 0 \\ dw &\longmapsto w\frac{dw}{w}, \quad \text{if } w \neq 0\ . \end{aligned}$$

This proves Lemma 1.3.2. □

Lemma 1.3.3. *Put*

$$\Theta_{\mathcal{C}/\mathcal{B}} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}). \tag{1.3.3}$$

Then $\Theta_{\mathcal{C}/\mathcal{B}}$ is an invertible $\mathcal{O}_{\mathcal{C}}$-sheaf and there is an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\omega_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}). \tag{1.3.3a}$$

Proof. By (1.3.2) it is easy to show that in a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$ we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} \frac{\partial}{\partial z}$$

and in a neighbourhood of a point $P \in \Sigma$ we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} \left(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w} \right). \tag{1.3.4}$$

By this fact and (1.3.1), we have the desired result. □

From the exact sequence (1.3.2) we obtain the following Corollary.

Corollary 1.3.4. *The following sequence*

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \Theta_{\mathcal{C}} \longrightarrow \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \longrightarrow 0$$

of $\mathcal{O}_{\mathcal{C}}$-sheaves is exact.

Lemma 1.3.5. *There exists an exact sequence*

$$0 \longrightarrow \Theta_{\mathcal{B}}(-\log D) \longrightarrow \Theta_{\mathcal{B}} \xrightarrow{t} \pi_* \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \longrightarrow 0, \tag{1.3.5}$$

where

$$\Theta_{\mathcal{B}}(-\log D) = \{\, v \in \Theta_{\mathcal{B}} \mid v(I_D) \subset I_D \,\}$$

and I_D is the sheaf of the defining ideal of D in $\mathcal{B}$.

Proof. First note that the sheaf $\Theta_{\mathcal{B}}(-\log D)$ is a sheaf of germs of vector fields on $\mathcal{B}$ tangent to D. Since $\pi\colon \mathcal{C} \to \mathcal{B}$ is a versal family, using the Kodaira-Spencer mapping and (1.2.1), for each point $s \in \mathcal{B}$ we have an exact sequence

$$0 \longrightarrow H^1\Big(C_s, \Theta_{C_s}\Big(-\sum_{j=1}^{N} s_j(s)\Big)\Big) \longrightarrow T_s\mathcal{B} \longrightarrow H^0\big(C_s, \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s})\big) \longrightarrow 0.$$

Each element of $H^1\big(C_s, \Theta_{C_s}\big(-\sum_{j=1}^{N} s_j(s)\big)\big)$ corresponds to a tangent vector of $\mathcal{B}$ at s preserving the singularities of C_s. Since $R^1\pi_* \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) = 0$ by the base change theorem and the argument of the proof of the following theorem, the sheaf version of the above sequence is also exact. □

Theorem 1.3.6. *Let $(\pi\colon \mathcal{C} \to \mathcal{B}; s_1, s_2, \ldots, s_N)$ a versal family of N-pointed stable curves of genus g. Then there exists an $\mathcal{O}_{\mathcal{B}}$-module isomorphism*

$$\rho\colon \Theta_{\mathcal{B}}(-\log D) \xrightarrow{\sim} R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \tag{1.3.6}$$

where we put $S_j = s_j(\mathcal{B})$ and $S = \sum_{j=1}^{N} S_j$.

Proof. Applying $\underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\cdot, \mathcal{O}_{\mathcal{C}})$ to the exact sequence (1.3.2), we obtain the following exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \Theta_{\mathcal{C}} \longrightarrow \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}\big(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}\big) \longrightarrow 0. \tag{1.3.7}$$

This exact sequence splits into the following short exact sequences

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \Theta_{\mathcal{C}} \xrightarrow{\kappa} \mathcal{M} \longrightarrow 0, \tag{1.3.8}$$

$$0 \longrightarrow \mathcal{M} \longrightarrow \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \longrightarrow 0. \tag{1.3.9}$$

Let $\mathcal{T}$ be the sheaf of germs of holomorphic vector fields on $\mathcal{C}$ preserving the n-th infinitesimal neighbourhoods. The sheaf $\mathcal{T}$ is given by

$$\mathcal{T} = \{\, v \in \Theta_{\mathcal{C}} \mid v(I_S) \subset I_S \,\}. \tag{1.3.10}$$

The sheaf $\mathcal{T}$ is an $\mathcal{O}_{\mathcal{C}}$-subsheaf of $\Theta_{\mathcal{C}}$ and coincides with $\Theta_{\mathcal{C}}$ outside $\bigcup_{j=1}^{N} S_j$. For a point $P \in S_j$ we let $(u_1, u_2, \ldots, u_M, z)$ be local coordinates of $\mathcal{C}$ with center P such that $(u_1, u_2, \ldots, u_M)$ are the coordinates of $\mathcal{B}$ with center $\pi(P)$ and such that S_j is defined by the equation $z = 0$ in a neighbourhood of P. Then, in a neighbourhood of P the sheaf $\mathcal{T}$ is generated by

$$z\frac{\partial}{\partial z}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_M}$$

as an $\mathcal{O}_{\mathcal{C}}$-sheaf. Hence $\mathcal{T}$ is locally free on $\mathcal{C}$.

Let us examine the exact sequences (1.3.8) and (1.3.9). Since the support of the sheaf $\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}})$ is in Σ, the sheaf $\mathcal{M}$ is equal to $\pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}}$ on $\mathcal{O}_{\mathcal{C}} \setminus \Sigma$. By using the above local coordinates of $\mathcal{C}$ with center $P \in S_j$, the restriction of the mapping κ in (1.3.8) to $\mathcal{T}$ in a neighbourhood of P is given by

$$a(u,z)z\frac{\partial}{\partial z} + \sum B_j(u,z)\frac{\partial}{\partial u_j} \longmapsto \sum B_j(u,z)\frac{\partial}{\partial u_j}.$$

Hence $\kappa\colon \mathcal{T} \to \mathcal{M}$ is surjective and its kernel is $\Theta_{\mathcal{C}/\mathcal{B}}(-(n+1)S)$ in a neighbourhood of P. On the other hand, on $\mathcal{B} \setminus \bigcup_{j=1}^N S_j$ the sheaf $\mathcal{T}$ is equal to $\Theta_{\mathcal{C}}$. Thus we have an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-S) \longrightarrow \mathcal{T} \longrightarrow \mathcal{M} \longrightarrow 0. \tag{1.3.8a}$$

¿From the exact sequence (1.3.8a) we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \xrightarrow{\tau} \pi_*\mathcal{T} \longrightarrow \pi_*\mathcal{M} \xrightarrow{\rho} R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \\ \longrightarrow R^1\pi_*\mathcal{T} \longrightarrow R^1\pi_*\mathcal{M} \longrightarrow 0. \end{aligned} \tag{1.3.11}$$

Put $\mathcal{B}_0 = \mathcal{B}\setminus D, \mathcal{C}_0 = \pi^{-1}(\mathcal{B}_0), \pi_0 = \pi|\mathcal{C}_0$. Then on $\mathcal{B}_0$, $\pi_{0*}\mathcal{M} = \Theta_{\mathcal{B}}$ and the homomorphism ρ is the Kodaira-Spencer mapping by Corollary 1.2.5. Since our family is versal, ρ is an isomorphism on $\mathcal{B}_0$. Therefore, the sheaf homomorphism τ in (1.3.11) is an isomorphism on $\mathcal{B}_0$. But on $\mathcal{B}_0$ we have $\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0$. Therefore, $\pi_*\mathcal{T} = 0$ on $\mathcal{B}_0$. As $\mathcal{T}$ is locally free, $\pi_*\mathcal{T}$ is torsion free, hence $\pi_*\mathcal{T} = 0$ on $\mathcal{B}$. This also implies

$$\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0 \tag{1.3.12}$$

on $\mathcal{B}$.

Next we show that ρ in (1.3.11) is an isomorphism. For that purpose it is enough to show that $R^1\pi_*\mathcal{T}$ is locally free. Because, if $R^1\pi_*\mathcal{T}$ is locally free, as ρ is isomorphic on $\mathcal{B}_0$, $\operatorname{Coker}\rho$ is a torsion subsheaf of $R^1\pi_*\mathcal{T}$, hence zero. By general cohomology theory of coherent sheaves,

$$\chi(\mathcal{T} \otimes \mathcal{O}_{C_s}) = \dim_{\mathbb{C}} H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) - \dim_{\mathbb{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$$

is independent of $s \in \mathcal{B}$, where $C_s = \pi^{-1}(s)$. (See, for example, [3].) Moreover, if $\dim_{\mathbb{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$ is, say k, independent of s, $R^1\pi_*\mathcal{T}$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module of rank k on $\mathcal{B}$, since we have $H^2(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$. Therefore, it is enough to show that $H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$ for all $s \in \mathcal{B}$.

Since C_s is a locally complete intersection in $\mathcal{C}$, we have an exact sequence

$$0 \longrightarrow \Theta_{C_s} \longrightarrow \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \longrightarrow \mathcal{O}_{C_s}(N) \longrightarrow \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{\mathcal{O}_{C_s}}, \mathcal{O}_{C_s}) \longrightarrow 0,$$

where N is the normal bundle of C_s in $\mathcal{C}$ which is a trivial bundle of rank $3g-3+N$. (See, for example, [1].) From this exact sequence we obtain two short exact sequences

$$\begin{aligned} 0 \longrightarrow \Theta_{C_s} \longrightarrow \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \longrightarrow M_s \longrightarrow 0, \\ 0 \longrightarrow M_s \longrightarrow \mathcal{O}_{C_s}(N) \longrightarrow \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega_{C_s}, \mathcal{O}_{C_s}) \longrightarrow 0. \end{aligned}$$

Similarly we have an exact sequence

$$0 \longrightarrow \Theta_{C_s}(-\sum_{j=1}^N Q_j) \longrightarrow \mathcal{T} \otimes \mathcal{O}_{C_s} \longrightarrow M_s \longrightarrow 0,$$

where $Q_j = s_j(s)$. This gives a long exact sequence

$$0 = H^0\Big(C_s, \Theta_{C_s}\Big(-\sum Q_j\Big)\Big) \longrightarrow H^0\Big(C_s, \mathcal{T}\otimes\mathcal{O}_{C_s}\Big) \longrightarrow H^0(C_s, M_s)$$
$$\xrightarrow{\rho} H^1\Big(C_s, \Theta_{C_s}\Big(-\sum Q_j\Big)\Big) \longrightarrow H^1\Big(C_s, \mathcal{T}\otimes\mathcal{O}_{C_s}\Big). \tag{1.3.13}$$

The cohomology group $H^0(C_s, M_s)$ parameterizes infinitesimal displacements of C_s in $\mathcal{C}$. (For the details see Tsuboi [37].) Since $\pi\colon \mathcal{C}\to\mathcal{B}$ is a versal family, infinitesimal displacements of C_s in $\mathcal{C}$ and infinitesimal deformations of C_s coincide. Hence the homomorphism ρ in (1.3.13) is an isomorphism. Hence we have $H^0(C_s, \mathcal{T}\otimes\mathcal{O}_{C_s}) = 0$.

Finally we show that $\pi_*\mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}}(-\log D)$. From (1.3.9) we obtain an exact sequence

$$0 \longrightarrow \pi_*\mathcal{M} \longrightarrow \Theta_{\mathcal{B}} \xrightarrow{t} \pi_*\left(\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}})\right).$$

The homomorphism t is the same as the one appearing in the exact sequence (1.3.5). Hence t is surjective. Therefore, by Lemma 1.3.5, $\pi_*\mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}}(-\log D)$. □

Remark 1.3.7. The homomorphism ρ in the above Theorem 1.3.6 is also called the *Kodaira-Spencer mapping*. The above proof shows that there exists an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}\Big(-\sum S_j\Big) \longrightarrow \mathcal{T} \longrightarrow \pi^{-1}\Theta_{\mathcal{B}}\otimes\mathcal{O}_{\mathcal{C}} \longrightarrow \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s}) \longrightarrow 0,$$

where $\mathcal{T}$ is a subsheaf of $\Theta_{\mathcal{C}}$ defined in (1.3.10). Choose a small Stein open set $\mathcal{U}\subset\mathcal{B}$ and a vector field $v\in H^0(\mathcal{U}, \Theta_{\mathcal{B}}(-\log D))$. Choose also a Stein open covering $\{\mathcal{U}_j\}_{j\in J}$ of $\pi^{-1}(\mathcal{U})$. Then v also defines an element $\pi^*v\in H^0(\mathcal{U}_j, \pi^{-1}\Theta_{\mathcal{B}}\otimes\mathcal{O}_{\mathcal{C}})$, whose image in $\underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s})$ is zero, since the tangent vector v is a direction of an infinitesimal deformation preserving singularities. Therefore, if $\mathcal{U}_j$ is small enough, we can find an element $v_j\in H^0(\mathcal{U}_j, \mathcal{T})$ which is mapped to π^*v. Then, we have

$$v_{ij} = v_j - v_i \in H^0\left(\mathcal{U}_i\cap\mathcal{U}_j, \Theta_{\mathcal{C}/\mathcal{B}}(-S)\right)$$

and $\{v_{ij}\}$ defines an element

$$[\{v_{ij}\}] \in H^1\left(\pi^{-1}(\mathcal{U}), \Theta_{\mathcal{C}/\mathcal{B}}(-S)\right).$$

The mapping

$$v \longmapsto [\{v_{ij}\}]$$

is nothing but the Kodaira-Spencer mapping ρ in Theorem 1.3.6.

Lemma 1.3.8. *Let $\mathfrak{F} = (\pi\colon \mathcal{C}\to\mathcal{B}; s_1, s_2, \ldots, s_N)$ be a versal family of N-pointed* smooth *curves of genus g. Let X, Y be a holomorphic vector field of $\mathcal{B}$ and*

$$\rho\colon \Theta_{\mathcal{B}} \longrightarrow R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S)$$

be the Kodaira-Spencer mapping. Then, we have

$$\rho([X,Y]) = [\rho(X), \rho(Y)] + X\rho(Y) - Y\rho(X),$$

where by putting

$$\rho(X) = \{\theta_{\lambda\mu}\}, \qquad \rho(Y) = \{\tau_{\lambda\mu}\}$$

$\rho([X,Y])$ is the cocycle $\{\vartheta_{\lambda\mu}\}$ defined as

$$\vartheta_{\lambda\mu} = [\theta_{\lambda\mu}, \tau_{\lambda\mu}]$$

and $X\rho(Y)$ is the cocycle $\{\tau'_{\lambda\mu}\}$ defined as

$$\tau'_{\lambda\mu} = X(\tau_{\lambda\mu}).$$

Proof. We may assume that $\mathcal{B}$ is small enough. Let $\{U_{\lambda\mu}\}$ be a sufficiently fine open covering of $\mathcal{C}$. Let $(z_1, \ldots, z_m)$ be coordinates of $\mathcal{B}$ and $(z_1, \ldots, z_m, \xi_\lambda)$ be a local coordinates of U_λ. Moreover, we may assume that for $j \in \{1, 2, \ldots, N\} \subset \Lambda$, U_j is a coordinate neighbourhood of $s_j(\mathcal{B})$ such that ξ_j induces an n-th infinitesimal neighbourhood η_j and that $s_j(\mathcal{B})$ is defined by $\xi_j = 0$. For $U_\lambda \cap U_\mu \neq \varnothing$, we have

$$\xi_\lambda = h_{\lambda\mu}(\xi_\mu, z_1, \ldots, z_m).$$

The vector field X induces an infinitesimal transformations of the coordinates $(z_1, \ldots, z_m)$:

$$(z_1, \ldots, z_m) \longmapsto (z_1 + \epsilon_1 a_1(z), \ldots, z_m + \epsilon_1 a_m(z)) \tag{1.3.14}$$

where ϵ_1 is a dual number, that is $\epsilon_1^2 = 0$ and

$$X = \sum_{j=1}^{m} a_i(z) \frac{\partial}{\partial z_i}.$$

For simplicity we express (1.3.14) as $z \mapsto z + \epsilon_1 X$. Since our family is versal, we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_1 X) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\mu, z)$$

where we put

$$\theta_{\lambda\mu} = \ell_{\lambda\mu} \frac{d}{d\xi_\mu},$$
$$\tau_{\lambda\mu} = m_{\lambda\mu} \frac{d}{d\xi_\mu}.$$

Similarly we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\mu, z),$$

where ϵ_2 is another dual number. Now let us calculate the result when we first deform infinitesimally in the direction of X and then deform infinitesimally in the direction of Y:

$$\begin{aligned} g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) &+ \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda + \epsilon_2 m_{\lambda\mu}, z + \epsilon_2 Y) \\ &= g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) \\ &\quad + \epsilon_1\epsilon_2 (Y(\ell_{\lambda\mu}(\xi_\lambda, z)) + m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}. \end{aligned}$$

If we deform in the opposite order, we have

$$\begin{aligned} g_{\lambda\mu}(\xi_\mu, z) &+ \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) \\ &\quad + \epsilon_1\epsilon_2 (X(m_{\lambda\mu}(\xi_\lambda, z)) + \ell_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}. \end{aligned}$$

Therefore, if we put

$$\rho([X, Y]) = \{\widetilde{\vartheta}_{\lambda\mu}\}$$

we have

$$\begin{aligned} \widetilde{\vartheta}_{\lambda\mu} &= \{\ell_{\lambda\mu} \frac{dm_{\lambda\mu}}{d\xi_\lambda} - m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda} + X(m_{\lambda\mu}) - Y(\ell_{\lambda\mu})\} \frac{d}{d\xi_\lambda} \\ &= [\theta_{\lambda\mu}, \tau_{\lambda\mu}] + X(\tau_{\lambda\mu}) - Y(\ell_{\lambda\mu}\} \frac{d}{d\xi_\lambda} \\ &= [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X)) \end{aligned}$$

□

1.4. Period Mappings and Bidifferentials.

Let R be a compact Riemann surface of genus g and $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ be a symplectic basis of $H_1(R, \mathbb{Z})$, that is,

$$\alpha_i \cdot \alpha_j = 0, \qquad \beta_i \cdot \beta_j = 0, \qquad \alpha_i \cdot \beta_j = \delta_{ij}.$$

A period matrix τ of the Riemann surface R is given by

$$\tau := \left(\int_{\beta_i} \omega_j\right) \cdot \left(\int_{\alpha_i} \omega_j\right)^{-1}, \tag{1.4.1}$$

where $\{\omega_1, \ldots, \omega_g\}$ is a basis of the holomorphic one-forms on R. Note that the period matrix τ is independent of the choice of the basis of holomorphic one-forms but depends on the choice of the symplectic bases. Let $\{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g\}$ be another symplectic basis of $H_1(R, \mathbb{Z})$. Then we have

$$\begin{pmatrix} \widetilde{\beta_1} \\ \vdots \\ \widetilde{\beta_g} \\ \widetilde{\alpha_1} \\ \vdots \\ \widetilde{\alpha_g} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \\ \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix},$$

where the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is in the group $Sp(g, \mathbb{Z})$ of symplectic integral matrices defined by

$$Sp(g, \mathbb{Z}) := \left\{ X \in M_{2g}(\mathbb{Z}) \;\middle|\; {}^tX \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} X = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \right\}.$$

Let $\widetilde{\tau}$ be a period matrix of the Riemann surface R defined by the symplectic basis $\{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g\}$. Then we have

$$\widetilde{\tau} = (A\tau + B)(C\tau + D)^{-1}.$$

A basis $\{\omega_1, \ldots, \omega_g\}$ of holomorphic one forms of R is called a *normalized basis* with respect to a symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ of $H_1(R, \mathbb{Z})$, if we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad \int_{\beta_i} \omega_j = \tau_{ij}, \quad \tau = (\tau_{ij}).$$

Let $\pi\colon \mathcal{R} \to \mathbb{T}$ be the universal family of compact Riemann surfaces of genus g over the Teichmüller space $\mathbb{T}$. For each point $t \in \mathbb{T}$, put $R_t = \pi^{-1}(t)$. There is a canonical isomorphism

$$H^1(R_t, \Theta_{R_t}) \simeq T_t\mathbb{T} \tag{1.4.2}$$

which is nothing but the one induced by the Kodaira-Spencer mapping. Taking the dual on both sides, we get a canonical isomorphism

$$T_t\mathbb{T}^* \simeq H^0(R_t, \omega_{R_t}^{\otimes 2}). \tag{1.4.3}$$

In the following we often identify the both sides of (1.4.3). Since $\mathbb{T}$ is simply connected, the sheaf $R_1\pi_*\mathbb{Z}$ is trivial on $\mathbb{T}$ and we find global sections

$$\{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_g, \widehat{\beta}_1, \ldots, \widehat{\beta}_g\}$$

of the sheaf $R_1\pi_*\mathbb{Z}$, such that for each point $t \in \mathbb{T}$,

$$\{\widehat{\alpha}_1(t), \ldots, \widehat{\alpha}_g(t), \widehat{\beta}_1(t), \ldots, \widehat{\beta}_g(t)\}$$

is a symplectic basis of $H_1(R_t, \mathbb{Z})$. The sheaf $\pi_*\omega_{\mathcal{R}/\mathbb{T}}$ is a free $\mathcal{O}_{\mathbb{T}}$-module of rank g. Let $\{\widehat{\omega}_1, \ldots, \widehat{\omega}_g\}$ be an $\mathcal{O}_{\mathbb{T}}$-free basis of $\pi_*\omega_{\mathcal{R}/\mathbb{T}}$. Then, the matrix

$$\Pi(t) := \left(\int_{\widehat{\beta}_i(t)} \widehat{\omega}_j(t)\right) \cdot \left(\int_{\widehat{\alpha}_i(t)} \widehat{\omega}_j(t)\right)^{-1} \tag{1.4.4}$$

defines a holomorphic mapping

$$\Pi\colon \mathbb{T} \longrightarrow \mathfrak{S}_g \tag{1.4.5}$$

from the Teichmüller space to the Siegel upper half plane of degree g. Moreover, there exists a group homomorphism

$$\Phi\colon \mathrm{Mod}_g \longrightarrow Sp(g, \mathbb{Z}) \tag{1.4.6}$$

of the mapping class group Mod_g of genus g to the symplectic group such that

$$\Pi(\gamma t) = (A_\gamma \Pi(t) + B_\gamma)(C_\gamma \Pi(t) + D_\gamma)^{-1}, \tag{1.4.7}$$

where

$$\Phi(\gamma) = \begin{pmatrix} A_\gamma & B_\gamma \\ B_\gamma & D_\gamma \end{pmatrix}. \tag{1.4.8}$$

For a compact Riemann surface R let $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ be a symplectic basis and $\{\omega_1, \ldots, \omega_g\}$ be a normalized basis of holomorphic one forms. In the following $(R, \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\})$ is called a *weakly marked Riemann surface*. Let $\tau \in \mathfrak{S}_g$ be the corresponding period matrix. We let $J(R)$ be a g-dimensional complex torus with period matrix (I_g, τ). Then, we can define a holomorphic mapping $j\colon R \to J(R)$ by

$$j(Q) := \left(\int_{P_0}^{Q} \omega_1, \ldots, \int_{P_0}^{Q} \omega_g\right), \tag{1.4.9}$$

where the base point $P_0 \in R$ is fixed once and for all. If we change the base point, the mapping j differs only by a translation of the complex torus $J(R)$. The mapping j is called the *Albanese* mapping and the complex torus $J(R)$ is usually called the Jacobian variety of the Riemann surface R.

The Jacobian variety $J(R)$ may be defined intrinsically by

$$J(R) = H^0(R, \omega_R)^*/H_1(R, \mathbb{Z}). \tag{1.4.10}$$

Precisely speaking, this is the definition of the Albanese variety of the Riemann surface R. Since we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}$$

for a normalized basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(R, \omega_R)$ we may regard $\{\alpha_1, \ldots, \alpha_g\}$ as the corresponding dual basis of $H^0(R, \omega_R)^*$. Then, the equality

$$\int_{\beta_i} \omega_j = \tau_{ij}$$

implies that

$$\beta_i = \sum_{j=1}^{g} \tau_{ij}\alpha_j.$$

Therefore, we can express $J(R)$ as a complex torus $\mathbb{C}^g/(I_g, \tau)$ defined above.

The Picard variety $\mathrm{Pic}^0(R)$ of the Riemann surface R is defined by

$$\mathrm{Pic}^0(R) := H^1(R, \mathcal{O}_R)/H^1(R, \mathbb{Z}). \tag{1.4.11}$$

Choose again a normalized basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(R, \omega_R)$. Then, $\{\overline{\omega}_1, \ldots, \overline{\omega}_g\}$ is a basis of $H^1(R, \mathcal{O}_R)$. By the Hodge decomposition theorem, we have an isomorphism

$$H^1(R, \mathbb{C}) \simeq H^1(R, \mathcal{O}_R) \oplus H^0(R, \omega_R).$$

Let $\{\alpha_1^*, \ldots, \alpha_g^*, \beta_1^*, \ldots, \beta_g^*\}$ be a basis of the cohomology group $H^1(R, \mathbb{Z})$ dual to the basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ of $H_1(R, \mathbb{Z})$. Since we may regard $H^1(R, \mathbb{Z})$ as a lattice of $H^1(R, \mathbb{C})$, by the above isomorphism we may express

$$\begin{aligned}
\alpha_i^* &= \sum_{j=1}^g a_{ij}\overline{\omega}_j + \sum_{j=1}^g \overline{a}_{ij}\omega_j, \\
\beta_i^* &= \sum_{j=1}^g b_{ij}\overline{\omega}_j + \sum_{j=1}^g \overline{b}_{ij}\omega_j.
\end{aligned}$$

Put

$$A = (a_{ij}) \quad B = (b_{ij}).$$

By the equalities

$$\begin{aligned}
\delta_{ik} = \langle \alpha_k, \alpha_i^* \rangle &= \sum_{j=1}^g a_{ij} \int_{\alpha_k} \overline{\omega}_j + \sum_{j=1}^g \int_{\alpha_k} \overline{a}_{ij}\omega_j \\
&= a_{ik} + \overline{a}_{ik}, \\
0 = \langle \beta_k, \alpha_i^* \rangle &= \sum_{j=1}^g a_{ij} \int_{\beta_k} \overline{\omega}_j + \sum_{j=1}^g \overline{a}_{ij} \int_{\beta_k} \omega_j \\
&= \sum_{j=1}^g a_{ij}\overline{\tau}_{jk} + \sum_{j=1}^g \overline{a}_{ij}\tau_{jk}.
\end{aligned}$$

and the similar equalities for β_i^*, we have

$$\begin{pmatrix} A & \overline{A} \\ B & \overline{B} \end{pmatrix} \begin{pmatrix} I_g & \overline{\tau} \\ I_g & \tau \end{pmatrix} = I_{2g}$$

Put

$$Y = (-2\pi\sqrt{-1}\,\mathrm{Im}\,\tau)^{-1}. \tag{1.4.12}$$

Note that Y is a symmetric matrix. We then have

$$\begin{pmatrix} I_g & \overline{\tau} \\ I_g & \tau \end{pmatrix}^{-1} = \begin{pmatrix} -\tau Y & -\overline{\tau}Y \\ Y & Y \end{pmatrix}.$$

Hence, the lattice induced by $H^1(R, \mathbb{Z})$ in $H^1(R, \mathcal{O}_R)$ is given by the row of the matrix

$$\begin{pmatrix} -\tau Y \\ Y \end{pmatrix},$$

where we use $\{\overline{\omega}_1, \ldots, \overline{\omega}_g\}$ as a basis of $H^1(R, \mathcal{O}_R)$. Hence, we may express $\mathrm{Pic}^0(R)$ as a complex torus $\mathbb{C}^g/(Y\tau, Y)$. Thus, there is an isomorphism

$$\lambda\colon J(R) \longrightarrow \mathrm{Pic}^0(R)$$

given by

$$\lambda \begin{bmatrix} z_1 \\ \vdots \\ z_g \end{bmatrix} = \left[Y \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \right]. \tag{1.4.13}$$

Note that the isomorphism λ is induced from a $\mathbb{C}$-linear isomorphism

$$\begin{aligned} H^0(R, \omega_R)^* &\longrightarrow H^1(R, \mathcal{O}_R), \\ \sum a_i \alpha_i &\longmapsto \sum_{ij} (y_{ij} a_j) \overline{\omega}_i, \end{aligned} \tag{1.4.14}$$

where $(y_{ij}) = Y$.

The Albanese and Picard varieties of a compact Riemann surface are dual to each other and they are principally polarized Abelian varieties. The above isomorphism $\lambda\colon J(R) \to \mathrm{Pic}^0(R)$ is a canonical one. The sheaf $\Theta_{J(R)}$ of holomorphic vector fields on the Jacobian variety is trivial and we have a canonical isomorphism

$$\Theta_{J(R)} \simeq T_0 J(R) \otimes \mathcal{O}_{J(R)}$$

Moreover, the Albanese mapping $j\colon R \to J(R)$ induces an isomorphism

$$H^1(J(R), \mathcal{O}) \simeq H^1(R, \mathcal{O})$$

by mapping $d\overline{z}_j \mapsto \overline{\omega}_j$. Therefore, we have a canonical isomorphism

$$\iota\colon H^1(J(R), \Theta_{J(R)}) \simeq T_0 J(R) \otimes H^1(R, \mathcal{O}_R).$$

where $T_0 J(R)$ is the tangent space of $J(R)$ at the origin. By the above isomorphism λ, we may identify $T_0 J(R)$ with $H^1(R, \mathcal{O}_R)$. Hence, we may rewrite

$$\iota\colon H^1(J(R), \Theta_{J(R)}) \simeq H^1(R, \mathcal{O}_R) \otimes H^1(R, \mathcal{O}_R). \tag{1.4.15}$$

Since $T_0 J(R)$ is identified with $H^0(R, \omega_R)^*$, by (1.4.14) the mapping ι can by (1.4.11) be written explicitly as

$$\iota\left(\frac{\partial}{\partial z_i} d\overline{z}_j\right) = \sum_{k=1}^{g} y_{ik} \overline{\omega}_k \otimes \overline{\omega}_j, \tag{1.4.16}$$

where the matrix $(y_{ij}) = Y$ is defined in (1.4.12). Put

$$\mathfrak{T} := \left\{ T = (t_{ij}) \;\middle|\; T \text{ is } g \times g \text{ complex matrix with } \det \begin{pmatrix} I_g & T \\ I_g & \overline{T} \end{pmatrix} \neq 0 \right\}$$

and we let act $\mathbb{Z}^{2g}$ act on $\mathfrak{T} \times \mathbb{C}^g$ by

$$\underline{n}\colon (T, z) \longrightarrow (T, z + \underline{n}_1 + \underline{n}_2 T), \quad \underline{n} = (\underline{n}_1, \underline{n}_2), \quad \underline{n}_i \in \mathbb{Z}^g.$$

The quotient space $\pi\colon \mathcal{T} = \mathfrak{T} \times \mathbb{C}^g/\mathbb{Z}^{2g} \to \mathfrak{T}$ is a versal family of complex tori. Let $\mu\colon \mathcal{A} \to \mathfrak{S}_g$ be the family of Abelian varieties over the Siegel upper half plane such that the fiber $A_\tau = \mu^{-1}(\tau)$

is the complex torus with period matrix (I_g, τ). There is a natural embedding $i\colon \mathfrak{S}_g \hookrightarrow \mathfrak{T}$ and the induced family $i^*\mathcal{T} \to \mathfrak{S}_g$ is isomorphic to the family $\mu\colon \mathcal{A} \to \mathfrak{S}_g$. We also call the mapping

$$i \circ \Pi\colon \mathbb{T} \longrightarrow \mathfrak{T}$$

the period mapping. At a point $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$ corresponding to the Riemann surface R with symplectic basis

$$\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\},$$

we may identify the tangent space $T_\tau\mathfrak{T}$ of $\mathfrak{T}$ at τ with

$$H^1(J(R), \Theta_{J(R)}),$$

by using the Kodaira-Spencer mapping. Hence, the period mapping $i \circ \Pi$ induces a natural mapping

$$H^1(R, \Theta_R) \longrightarrow H^1(J(R), \Theta_{J(R)}) = H^1(R, \mathcal{O}_R) \otimes H^1(R, \mathcal{O}_R),$$

where we use the isomorphism ι (1.4.14). The dual of this mapping is given by the natural multiplication map

$$\begin{aligned} H^0(R, \omega_R) \otimes H^0(R, \omega_R) &\longrightarrow H^0(R, \omega_R^{\otimes 2}), \\ \nu \otimes \mu &\longmapsto \quad \nu\mu. \end{aligned} \tag{1.4.17}$$

Let us calculate the Kodaira-Spencer class of a tangent vector $\frac{\partial}{\partial t_{ij}}$ at a point $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$. Let $\mathcal{U}$ be an open neighbourhood of the point τ_0 and we let $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ be a small open covering of $\mu^{-1}(\mathcal{U})$. We choose local coordinates $(t_{ij}, z_\lambda^1, \ldots, z_\lambda^g)$, $1 \leqslant i \leqslant j \leqslant g$ in such a way that $(z_\lambda^1, \ldots, z_\lambda^g)$ are deduced from the global affine coordinates $(z_1, \ldots, z_g)$ of the vector space $\mathbb{C}^g$. Thus, if $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \varnothing$, then we have

$$z_\lambda^i - z_\mu^i = m_{\lambda\mu}^i + \sum_{k=1}^{g} n_{\lambda\mu}^k t_{ik}, \tag{1.4.18}$$

where $m_{\lambda\mu}^i$, $n_{\lambda\mu}^i \in \mathbb{R}^g$, $i = 1, 2, \ldots, g$. Therefore, the Kodaira-Spencer class $\rho(\frac{\partial}{\partial t_{ij}})$ is given by a cocycle $\theta_{\lambda\mu}$ defined by

$$\theta_{\lambda\mu} = n_{\lambda\mu}^j \frac{\partial}{\partial z_\lambda^i}. \tag{1.4.19}$$

Now put

$$\begin{pmatrix} \psi_\lambda^1 \\ \vdots \\ \psi_\lambda^g \end{pmatrix} = Y \begin{pmatrix} \overline{z}_\lambda^1 - z_\lambda^1 \\ \vdots \\ \overline{z}_\lambda^g - z_\lambda^g \end{pmatrix}.$$

where $Y = (-2\sqrt{-1}\,\mathrm{Im}(t_{ij}))^{-1}$. Then we have

$$\psi_\mu^j \frac{\partial}{\partial z_\mu^i} - \psi_\lambda^j \frac{\partial}{\partial z_\lambda^i} = \theta_{\lambda\mu}.$$

Thus

$$\overline{\partial}\left(\psi_\mu^j \frac{\partial}{\partial z_\mu^i}\right) = \overline{\partial}\left(\psi_\lambda^j \frac{\partial}{\partial z_\lambda^i}\right)$$

is a global form and is expressed as

$$\sum_{k=1}^{g} y_{jk} \frac{\partial}{\partial z_i} d\overline{z}_k. \tag{1.4.20}$$

The Dolbeault cohomology class of the form (1.4.20) is the Kodaira-Spencer class $\rho(\frac{\partial}{\partial t_{ij}})$ and by (1.4.14) we have

$$\iota\left(\rho\frac{\partial}{\partial t_{ij}}\right)=\sum_{\ell=1}^{g}\sum_{k=1}^{g}\{(y_{i\ell}\overline{\omega}_\ell)\otimes(y_{jk}\overline{\omega}_k)\}.$$

Hence, by (1.4.14) the dual of this class in $H^0(R,\omega_R)\otimes H^0(R,\omega_R)$ is $\omega_i\otimes\omega_j$. Hence, by (1.4.18) we conclude

$$\Pi^*(d\tau_{ij})=\Pi^*\circ i^*(dt_{ij})=\omega_i\omega_j, \tag{1.4.21}$$

where we use the identification (1.4.3). Thus we have proved the following proposition.

Proposition 1.4.1. *Under the above assumption and notation, we have*

$$\begin{aligned}\Pi^*\Big(\sum_{i<j}\frac{\partial}{\partial\tau_{ij}}(\log\det(C_\gamma\tau+D_\gamma))d\tau_{ij}\Big) \\ =\sum_{i<j}\omega_i\omega_j\Pi^*\left(\frac{\partial}{\partial\tau_{ij}}\log\det(C_\gamma\tau+D_\gamma)\right).\end{aligned} \tag{1.4.22}$$

Next let us consider a symmetric bidifferential form on a compact Riemann surface and a stable curve. First we recall basic facts on theta functions. For a point $\tau\in\mathfrak{S}_g$ the theta function $\theta(\tau,z)$ is defined as

$$\theta(\tau,z):=\sum_{m\in\mathbb{Z}^g}\exp\left\{2\pi\sqrt{-1}(\frac{1}{2}{}^t m\tau m+{}^t mz)\right\}.$$

The theta function $\theta(\tau,z)$ can be considered as a holomorphic section of the line bundle which defines a principal polarization of the Abelian variety $A_\tau=\mathbb{C}/(I_g,\tau)$. For $e\in\mathbb{C}^g$ by $[e]$ we denote the corresponding point of A_τ. The zero of the theta function $\theta(\tau,z)$ defines a divisor on A_τ called the *theta divisor* and is denoted as Θ.

The theta function $\theta[e](\tau,z)$ with characteristic $e=\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}$, $\delta,\epsilon\in\mathbb{R}^g$ is defined as

$$\theta[e](\tau,z):=\sum_{m\in\mathbb{Z}^g}\exp 2\pi\sqrt{-1}\left\{\frac{1}{2}{}^t(m+\delta)\tau(m+\delta)+{}^t(m+\delta)(z+\epsilon))\right\}.$$

Let R be a compact Riemann surface of genus g. Let us fix a symplectic basis of $H_1(R,\mathbb{Z})$ and let $\tau\in\mathfrak{S}_g$ be the corresponding period matrix. Then the Abelian variety A_τ is the Jacobian variety $J(R)$ of R. Let $\mathrm{Div}^d(R)$ be the group of divisors of degree d on the compact Riemann surface R. We can extend the mapping $j\colon R\to J(R)$ to a mapping $j\colon\mathrm{Div}^d(R)\to J(R)$ by putting

$$j\Big(\sum_{i=1}^{m+d}P_i-\sum_{k=1}^{m}Q_k\Big):=\sum_{i=1}^{m+d}j(P_i)-\sum_{k=1}^{m}j(Q_k).$$

Note that for two divisors $D,D'\in\mathrm{Div}^d(R)$, $j(D)=j(D')$ if and only if D and D' are linearly equivalent, that is, there is a meromorphic function f on R such that $D=D'+(f)$. Let $\mathrm{Pic}^d(R)$ be the set of isomorphism classes of line bundles of degree d on the compact Riemann surface R. Then, by Abel's theorem, the mapping $j\colon\mathrm{Div}^d(R)\to J(R)$ factors through $j\colon\mathrm{Pic}^d(R)\to J(R)$, where we identify $\mathrm{Pic}^d(R)$ with the quotient $\mathrm{Div}^d(R)/\sim$ by the linear equivalence. Moreover, for any divisor D on R by $[D]$ we denote the corresponding line bundle on R. Also in the following we always identify $\mathrm{Pic}^0(R)$ with $J(R)$ via the canonical isomorphism λ. The following Theorem due to Riemann is the most fundamental in the theory of theta functions associated with a Riemann surface.

Theorem 1.4.2. *There is a divisor* $\Delta \in \mathrm{Div}^{g-1}(R)$ *with*

$$[2\Delta] = \omega_R, \tag{1.4.23}$$

such that for any point $P \in R$ *and* $e \in \mathbb{R}^g$ *we have:*

(1) *If* $\theta(\tau, e) \neq 0$, *then the divisor* D *of* R *defined by the equation in the variable* Q

$$\theta\left(\tau, j(Q) - j(P) - e\right) = 0$$

is an effective divisor of degree g *with*

$$H^1(R, \mathcal{O}_R(D)) = 0$$

and

$$[e] = j(D - P - \Delta).$$

(2) *If* $\theta(\tau, e) = 0$, *then for some effective divisor* E *of degree* $g - 1$ *we have*

$$[e] = j(E - \Delta).$$

Moreover, $\dim_{\mathbb{C}} H^1(R, \mathcal{O}_R(E))$ *is the multiplicity of the theta divisor* Θ *at the point* $[e]$ *and it is the smallest integer* d *such that*

$$\theta(B - A - e) \equiv 0$$

for all effective divisor A, B *of degree* $\leqslant d - 1$.

Put

$$L_0 = [\Delta]. \tag{1.4.24}$$

Hence, we have

$$L_0^{\otimes 2} = \omega_R. \tag{1.4.25}$$

For a half period (i.e. a two torsion point) $\alpha \in J(C)$, that is, $2\alpha = 0$ in $J(R)$, we let L_α be the line bundle on the Riemann surface R such that $L_\alpha \otimes L_0^{-1}$ is the half period α. Hence, in particular

$$L_\alpha^{\otimes 2} = \omega_R. \tag{1.4.26}$$

Corollary 1.4.3. *We have*

$$h^0(R, L_\alpha) = h^1(R, L_\alpha) = \mathrm{mult}_\alpha\, \Theta \tag{1.4.27}$$

and $h^0(R, L_\alpha)$ *is even for* α *even and odd for* α *odd. Where a half period* α *is called odd (resp. even), if we have* $\alpha = \vec{a}/2 + \vec{b}\tau/2$ *and* $\vec{a}^t\vec{b}$ *is odd (resp. even).*

For the proofs of Theorem 1.4.2 and Corollary 1.4.3 we refer the reader to [11]. Let α be a non-singular odd half period, that is, $\mathrm{mult}_\alpha\, \Theta = 1$. Then, by the above corollary, the line bundle L_α has a non-zero global section h_α satisfying

$$h_\alpha^2 = \sum_{i=1}^{g} \frac{\partial \theta[\alpha]}{\partial z_i}(\tau, 0)\omega_i, \tag{1.4.28}$$

where $\{\omega_1, \ldots, \omega_g\}$ be the normalized basis of holomorphic one forms and we use (1.4.26). The *prime form* $E(P, Q)$ of the Riemann surface R is defined as

$$E(P, Q) := \frac{\theta[\alpha](\tau, j(P - Q))}{h_\alpha(P) h_\alpha(Q)} \tag{1.4.29}$$

for points P, $Q \in R$. Note that the prime form is independent of the choice of α and is a holomorphic section of the line bundle $\pi_1 L_0^{-1} \otimes \pi_2 L_0^{-1} \otimes \delta^*(\Theta)$ on $R \times R$ where π_i is the projection of $R \times R$ to the i-th factor and $\delta\colon R \times R \to J(R)$ is the mapping sending $(P, Q) \in R \times R$ to $j(Q - P)$. Moreover, we have

$$E(P, Q) = -E(Q, P) \tag{1.4.30}$$

and $E(P, Q)$ vanishes to the first order along the diagonal in $R \times R$ and does not vanish outside the diagonal. Let x (respectively, y) be a local coordinate of R with center P (respectively, q) and by abuse of notation let us write

$$h_\alpha(P) = h_\alpha(x)\sqrt{dx}$$
$$h_\alpha(Q) = h_\alpha(y)\sqrt{dy}.$$

Then the prime form $E(P, Q)$ is expressed as

$$E(P, Q) = E(x, y)(\sqrt{dy})^{-1}(\sqrt{dy})^{-1}.$$

Now let us define the bidifferential $\omega(x, y)dxdy$ by

$$\omega(x, y)dxdy := \frac{\partial^2}{\partial x \partial y} \log E(x, y)dxdy. \tag{1.4.31}$$

Then, $\omega(x, y)dxdy$ is a meromorphic bidifferential on $R \times R$ holomorphic outside the diagonal and has a pole of order two along the diagonal. That is, $\omega(x, y)dxdy \in H^0(R \times R, p_1^*\omega_R \otimes p_2^*\omega_R(2\Delta))$ Moreover, by (1.4.30) we have

$$\omega(x, y) = \omega(y, x). \tag{1.4.32}$$

Proposition 1.4.4. *For $f \in \mathbb{C}^g$ such that $[f] \in J(R)$ is a non-singular point of the theta divisor Θ, we have that*

$$\omega(x, y)dxdy = \frac{\partial^2 \log \theta}{\partial x \partial y}(\tau, j(x) - j(y) - f)dxdy.$$

For a proof, see [11].

For a symmetric bidifferential $\widehat{\omega}(x, y)dxdy$ on $R \times R$ holomorphic outside the diagonal, we always have an expansion at the diagonal

$$\widehat{\omega}(x, y)dxdy = \Big(\sum_{k=k_0}^{2} \frac{c_{-k}}{(x - y)^k} + \text{holomorphic}\Big).$$

The coefficient c_{-2} is independent of the choice of local coordinates and called the *biresidue* and written as $\mathrm{Res}^2(\widehat{\omega})$.

In the following by a meromorphic symmetric bidifferential $\omega(x, y)dxdy$ of a Riemann surface R we mean that $\omega(y, x) = \omega(x, y)$, ω is holomorphic on $R \times R$ outside the diagonal and has a pole of order 2 along the diagonal.

Lemma 1.4.5. *Let $\omega(x, y)dxdy$ and $\widetilde{\omega}(x, y)dxdy$ be symmetric bilinear differentials on $R \times R$ holomorphic outside the diagonal such that they have poles of order 2 at the diagonal with*

$$\mathrm{res}^2(\omega) = \mathrm{res}^2(\widehat{\omega})$$

Then the difference

$$\Omega(x, y)dxdy = \omega(x, y)dxdy - \widetilde{\omega}(x, y)dxdy$$

is a holomorphic symmetric bidifferentials on $R \times R$.

The space of symmetric bilinear differentials on $R \times R$ having pole of order two at the diagonal and holomorphic outside the diagonal is written as

$$H^0(R \times R, p_1^* \omega_R \otimes p_2^* \omega_R(2\Delta))^{\langle \iota \rangle}$$

where ι is the automorphism of $R \times R$ which interchanges the two factors. Note that

$$\omega_{R \times R} = p_1^* \omega_R \otimes p_2^* \omega_R$$

Let us define a *projective connection* $S_\omega(z)dz^2$ attached to a symmetric bidifferential ω with biresidue 1 of a compact Riemann surface R by

$$S_\omega(z)dz^2 = 6 \lim_{\substack{x \to z \\ y \to z}} \left(\omega(x,y)dxdy - \frac{dxdy}{(x-y)^2} \right).$$

The projective connection $S_\omega(z)dz^2$ does depend on choice of local coordinate z.

Theorem 1.4.6.

$$S_\omega(z)dz^2 = S_\omega(w)dw^2 - \{z;w\}dw^2$$

where $\{z;w\}$ is the Schwartian derivative defined by

$$\{z;w\} = \frac{d^3z}{dw^3} \Big/ \frac{dz}{dw} - \frac{3}{2} \left(\frac{d^2z}{dw^2} \Big/ \frac{dz}{dw} \right)^2.$$

Proof. Put

$$H(x,y)dxdy = \omega(x,y)dxdy - \frac{dxdy}{(x-y)^2}.$$

Let $g(u)$ be a holomorphic function in u. We have

$$\begin{aligned} H(x,y)dxdy &= \omega(g(u),g(v))g'(u)g'(v)dudv - \frac{g'(u)g'(v)dudv}{(g(u)-g(v))^2} \\ &= H(u,v)dudv - \left\{ \frac{g'(u)g'(v)dudv}{(g(u)-g(v))^2} - \frac{dudv}{(u-v)^2} \right\}. \end{aligned}$$

Hence, we have

$$S_\omega(z)dz^2 = S_\omega(w)dw^2 + 6 \lim_{\substack{x \to z \\ y \to z}} \left\{ \frac{g'(u)g'(v)}{(g(u)-g(v))^2} - \frac{1}{(u-v)^2} \right\} dw^2,$$

where $z = g(w)$. Let us calculate the last term of the above equality. Put

$$u_1 = u - w, \quad v_1 = v - w.$$

We have the Taylor expansion at $(u_1, v_1) = (0,0)$

$$\begin{aligned} \frac{g(u)-g(v)}{u-v} &= g'(w) + \frac{1}{2}g''(w)(u_1+v_1) \\ &\quad + \frac{1}{6}g'''(w)(u_1^2 + u_1v_1 + v_1^2) + \cdots. \end{aligned} \tag{1.4.33}$$

Put

$$F(u,v) = \log \left(\frac{g(u)-g(v)}{u-v} \right).$$

Then

$$F(w,w) = \log g'(w).$$

Moreover, by a direct calculation we have

$$\frac{\partial^2 F(u,v)}{\partial u \partial v} = \frac{g'(u)g'(v)}{(g(u)-g(v))^2} - \frac{1}{(u-v)^2}.$$

On the other hand, by (1.4.33) the Taylor expansion of $F(u_1+w, v_1+w)$ at $(u_1,v_1)=(0,0)$ has the form

$$\begin{aligned} F(u_1+w, v_1+w) = \log g'(w) &- \frac{1}{2}\frac{g''(w)}{g'(w)}(u_1+v_1) \\ &- \left\{\frac{1}{6}\frac{g'''(w)}{g'(w)} - \frac{1}{8}(\frac{g''(w)}{g'(w)})^2\right\}(u_1^2+v_1^2) \\ &- \left\{\frac{1}{6}\frac{g'''(w)}{g'(w)} - \frac{1}{4}(\frac{g''(w)}{g'(w)})^2\right\}u_1v_1 + \cdots. \end{aligned}$$

Hence, we conclude

$$\begin{aligned} 6 \lim_{\substack{x\to z \\ y\to z}} &\left\{\frac{g'(u)g'(v)}{(g(u)-g(v))^2} - \frac{1}{(u-v)^2}\right\} dw^2 \\ &= 6 \lim_{(u_1,v_1)\to(0,0)} \frac{\partial^2 F(u,v)}{\partial u \partial v} dw^2 \\ &= -\left\{\frac{g'''(w)}{g'(w)} - \frac{3}{2}(\frac{g''(w)}{g'(w)})^2\right\} dw^2 \\ &= -\{g;w\}dw^2 \end{aligned} \tag{1.4.34}$$

This proves the theorem. □

2. Affine Lie Algebras and Integrable Highest Weight Representations

2.1. Affine Lie Algebras and Integrable Highest Weight Modules.

In this section we recall basic facts on integrable highest weight representations of affine Lie algebras. For the details of this theory we refer the reader to Kac's book [19].

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbb{C}$ and $\mathfrak{h}$ its Cartan subalgebra. By Δ we denote the root system of $(\mathfrak{g},\mathfrak{h})$. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

Let $\mathfrak{h}^*_{\mathbb{R}}$ be the linear span of Δ over $\mathbb{R}$. Fix a lexicographic ordering of $\mathfrak{h}^*_{\mathbb{R}}$ once and for all. This gives the decomposition $\Delta = \Delta_+ \sqcup \Delta_-$ of the root system into the positive roots and the negative roots. Put $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, the linear span of Δ over $\mathbb{C}$. Let $(\cdot,\cdot)$ be a constant multiple of the Cartan-Killing form of the simple Lie algebra $\mathfrak{g}$. For each element of $\lambda \in \mathfrak{h}$, there exists a unique element $H_\lambda \in \mathfrak{h}^*$ such that

$$\lambda(H) = (H_\lambda, H)$$

for all $H \in \mathfrak{h}$. For $\alpha \in \Delta$, H_α is called the *root vector* corresponding to the root α.

On $\mathfrak{h}^*$ we introduce an inner product by

$$(\lambda,\mu) = (H_\lambda, H_\mu). \tag{2.1.1}$$

Let us normalize the inner product $(\cdot,\cdot)$ and call it the normalized Cartan-Killing form. Let θ be the highest (or longest) root, that is,

$$\theta \in \Delta_+, \quad \theta + \alpha_i \notin \Delta$$

for all simple roots α_i. For example, the highest root of the simple Lie algebra of type A_ℓ is given by

$$\theta = \sum_{i=1}^{\ell} \alpha_i$$

and the highest root of the simple Lie algebra of type D_ℓ is given by

$$\alpha_1 + 2\sum_{i=2}^{\ell-2} \alpha_i + \alpha_{\ell-1} + \alpha_\ell.$$

The normalized Cartan-Killing form is given by the condition

$$(\theta, \theta) = 2. \tag{2.1.2}$$

Note that the Cartan-Killing form has the following property.

$$([X,Y],Z) + (Y,[X,Z]) = 0. \tag{2.1.3}$$

Example 2.1.1. The simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ consists of traceless 2×2 matrices. Put

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These matrices enjoy the following commutation relations.

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

Then, $\mathfrak{h}$ is spanned by H and we have $\Delta = \{\alpha, -\alpha\}$ where $\alpha \in \mathfrak{h}^*$ is defined by $\alpha(H) = 2$. $\mathfrak{g}_\alpha$ is spanned by E and $\mathfrak{g}_{-\alpha}$ is spanned by F. The normalized Cartan-Killing form is given by

$$(X,Y) = Tr(XY), \quad X,Y \in \mathfrak{g}.$$

By $\mathbb{C}[[\xi]]$ and $\mathbb{C}((\xi))$ we mean the ring of formal power series in ξ and the field of formal Laurent power series in ξ, respectively.

Definition 2.1.2. The *affine Lie algebra* $\widehat{\mathfrak{g}}$ over $\mathbb{C}((\xi))$ associated with $\mathfrak{g}$ is defined to be

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c,$$

where c is an element of the center of $\widehat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$\begin{aligned}[X \otimes f(\xi), Y \otimes g(\xi)] = \\ [X,Y] \otimes f(\xi)g(\xi) + c \cdot (X,Y) \operatorname*{Res}_{\xi=0}(g(\xi)df(\xi))\end{aligned} \tag{2.1.4}$$

for

$$X, Y \in \mathfrak{g}, f(\xi), g(\xi) \in \mathbb{C}((\xi)).$$

Put

$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi, \quad \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[\xi^{-1}]\xi^{-1}. \tag{2.1.5}$$

We regard $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ as Lie subalgebras of $\widehat{\mathfrak{g}}$. Also in the following we often identify $\mathfrak{g}$ and $\mathfrak{g} \otimes 1$, so that we may regard $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$. We have a decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_-. \tag{2.1.6}$$

In the sequal we use the following nation freely

$$\begin{aligned} X(n) &:= X \otimes \xi^n, \quad X \in \mathfrak{g}, \quad n \in \mathbb{Z}, \\ X &= X(0) = X \otimes 1. \end{aligned}$$

Remark 2.1.3. Note that usually the affine Lie algebra is defined by using $\mathbb{C}[\xi, \xi^{-1}]$. Namely, put

$$\mathfrak{g}_{aff} := \mathfrak{g} \otimes \mathbb{C}[\xi, \xi^{-1}] \oplus \mathbb{C} \cdot c$$

with commutation relation (2.1.4). Put

$$\mathfrak{g}_+ = \mathfrak{g} \otimes \mathbb{C}[\xi]\xi.$$

This is a Lie subalgebra and we have

$$\mathfrak{g}_{aff} = \mathfrak{g}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c \oplus \widehat{\mathfrak{g}}_-.$$

Moreover, the Lie algebra $\widehat{\mathfrak{g}}_+$ is contained in the ξ-adic completion of $\mathfrak{g}_+$.

Let us define an endomorphism D of the vector space $\widehat{\mathfrak{g}}$ by

$$\begin{aligned} D(X \otimes f(\xi)) = X \otimes \xi f'(\xi), \quad X \in \mathfrak{g}, f(\xi) \in \mathbb{C}((\xi)), \\ D(c) = 0. \end{aligned} \tag{2.1.7}$$

That is, D is induced from a derivation $\xi \frac{d}{d\xi}$ on $\mathbb{C}((\xi))$. Then, by (2.1.4) we have

$$D([X \otimes f(\xi), Y \otimes g(\xi)]) = [D(X \otimes f(\xi)), Y \otimes g(\xi)] + [X \otimes f(\xi), D(Y \otimes g(\xi)].$$

Hence, D is a derivation of the Lie algebra $\widehat{\mathfrak{g}}$. Let

$$\widehat{\mathfrak{g}}^e := \widehat{\mathfrak{g}} \times \mathbb{C} \cdot D \tag{2.1.8}$$

be the semi-direct product of the Lie algebra $\widehat{\mathfrak{g}}$ and a one-dimensional Lie algebra $\mathbb{C} \cdot D$. Put

$$\widehat{\mathfrak{h}}^e = (\mathfrak{h}, 0) + \mathbb{C} \cdot (c, 0) + \mathbb{C} \cdot (0, D)$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ regarded as a subalgebra of $\widehat{\mathfrak{g}}$. Then $\widehat{\mathfrak{h}}^e$ is an Abelian subalgebra of $\widehat{\mathfrak{g}}^e$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the simple positive roots of $\mathfrak{g}$ with respect to the lexicographic ordering of $\mathfrak{h}^*_{\mathbb{R}}$. Let

$$\theta = \sum_{i=1}^{k} m_i \alpha_i \tag{2.1.9}$$

be the highest root of $\mathfrak{g}$. Let us defin elements $a_i \in (\widehat{\mathfrak{h}}^e)^*$, $i = 0, 1, 2, \ldots, k$ as follows. For $i = 1, 2 \ldots, k$, a_i is defined by

$$[h, X_{\alpha_i} \otimes 1] = a_i(h) X_{\alpha_i} \otimes 1,$$

for any $h \in \widehat{\mathfrak{h}}^e$, where $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$. The element a_0 is defined by

$$[h, X_{-\theta} \otimes \xi] = a_0(h) X_{-\theta} \otimes \xi,$$

for any element $h \in \widehat{\mathfrak{h}}^e$ and $X_{-\theta} \in \mathfrak{g}_{-\theta}$.

For each root $\alpha \in \Delta$ of $\mathfrak{g}$ with $\alpha = \sum_{i=1}^{k} r_i \alpha_i$, put

$$a(\alpha) := \sum_{i=1}^{k} r_i a_i. \tag{2.1.10}$$

Also put

$$\delta := \sum_{i=1}^{k} m_i a_i + a_0 \tag{2.1.11}$$

where the number m_i is defined by (2.1.9). Finally put

$$\Delta_I(\widehat{\mathfrak{g}}) := \{n\delta\}_{n \in \mathbb{Z} \setminus \{0\}}, \tag{2.1.12}$$

$$\Delta_W(\widehat{\mathfrak{g}}) := \{a(\alpha) + n\delta\}_{\alpha \in \Delta, n \in \mathbb{Z}}. \tag{2.1.13}$$

Each element of $\Delta_I(\widehat{\mathfrak{g}})$ (respectively, $\Delta_W(\widehat{\mathfrak{g}})$) is called an imaginary root (respectively, a Weyl root or a real root) of the affine Lie algebra $\widehat{\mathfrak{g}}$. Then, we have a root space decomposition of the affine Lie algebra $\mathfrak{g}_{aff}$

$$\mathfrak{g}_{aff} := \widehat{\mathfrak{h}} \bigoplus \coprod_{n\delta \in \Delta_I(\widehat{\mathfrak{g}})} \mathfrak{g}_{n\delta} \bigoplus \coprod_{a \in \Delta_W(\widehat{\mathfrak{g}})} \mathfrak{g}_a,$$

where $\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C} \cdot c$ and the affine Lie algebra $\widehat{\mathfrak{g}}$ is contained in the ξ-adic completion of $\mathfrak{g}_{aff}$. Note that

$$\mathfrak{g}_a = \mathfrak{g}_\alpha \otimes \xi^n, \quad a = a(\alpha) + n\delta, \alpha \in \Delta, n \in \mathbb{Z} \setminus \{0\},$$
$$\mathfrak{g}_a = \mathfrak{h} \otimes \xi^n, \quad a = n\delta, n \in \mathbb{Z} \setminus \{0\}.$$

Let us recall briefly the representation theory of a simple Lie algebra $\mathfrak{g}$. An irreducible left $\mathfrak{g}$-module V_λ is called a *highest weight module with highest weight* λ, if there exists a non-zero vector $e \in V_\lambda$ (called a highest weight vector) such that

$$He = \lambda(H)e, \quad Xe = 0 \quad \text{for all } H \in h, \quad X \in \mathfrak{g}_\alpha, \quad \alpha \in \Delta_+.$$

It is well-known that a finite dimensional irreducible left $\mathfrak{g}$-module is a highest weight module and two irreducible left $\mathfrak{g}$-modules are isomorphic if and only if they have the same highest weight. A weight $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is called an integral weight, if

$$2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z},$$

for any $\alpha \in \Delta$. A weight $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is called a dominant weight, if

$$w(\lambda) \leqslant \lambda,$$

for any element w of the Weyl group W of $\mathfrak{g}$. By P_+ we denote the set of *dominant integral weights* of $\mathfrak{g}$. A weight λ is the highest weight of an irreducible left $\mathfrak{g}$-module if and only if $\lambda \in P_+$.

Let us fix a positive integer ℓ (called the *level*) and put

$$P_\ell = \{\, \lambda \in P_+ \mid 0 \leq (\theta, \lambda) \leq \ell \,\}.$$

For each element $\lambda \in P_\ell$ we shall define the Verma module $\mathcal{M}_\lambda$ as follows. Put

$$\widehat{\mathfrak{p}}_+ := \widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c.$$

Then $\widehat{\mathfrak{p}}_+$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Let V_λ is the irreducible left $\mathfrak{g}$-module of highest weight λ. The action of $\widehat{\mathfrak{p}_+}$ on V_λ is defined as

$$cv = \ell v, \quad \text{for all } v \in V_\lambda,$$
$$av = 0, \quad \text{for all } a \in \widehat{\mathfrak{g}}_+ \text{ and } v \in V_\lambda.$$

Put

$$\mathcal{M}_\lambda := U(\widehat{\mathfrak{g}}) \otimes_{\widehat{\mathfrak{p}}_+} V_\lambda. \tag{2.1.14}$$

Then $\mathcal{M}_\lambda$ is a left $\widehat{\mathfrak{g}}$-module and it is called a *Verma module*. The Verma module $\mathcal{M}_\lambda$ is not irreducible. It contains the maximal proper submodule $\mathcal{J}_\lambda$. The quotient module $\mathcal{H}_\lambda := \mathcal{M}_\lambda / \mathcal{J}_\lambda$ has the following properties.

Theorem 2.1.4. *For each $\lambda \in P_\ell$, the left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ is the unique left $\widehat{\mathfrak{g}}$-module (called the* integrable highest weight $\widehat{\mathfrak{g}}$-module*) satisfying the following properties.*

(1) $V_\lambda = \{\, |v\rangle \in \mathcal{H}_\lambda \mid \widehat{\mathfrak{g}}_+|v\rangle = 0 \,\}$ *is the irreducible left $\mathfrak{g}$-module with highest weight λ.*
(2) *The central element c acts on $\mathcal{H}_\lambda$ as $\ell \cdot \mathrm{id}$.*
(3) *$\mathcal{H}_\lambda$ is generated by V_λ over $\widehat{\mathfrak{g}}_-$ with only one relation*

$$(X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle = 0, \tag{2.1.15}$$

where $X_\theta \in \mathfrak{g}$ is the element corresponding to the maximal root θ and $|\lambda\rangle \in V_\lambda$ is the highest weight vector.

The theorem says that the maximal proper submodule $\mathcal{J}_\lambda$ is given by

$$\mathcal{J}_\lambda = U(\widehat{\mathfrak{p}}_-)|J_\lambda\rangle, \tag{2.1.16}$$

where we put

$$|J_\lambda\rangle = (X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle. \tag{2.1.17}$$

For the details see Kac [19, (10.4.6)].

Similarly we have the integrable lowest weight right $\widehat{\mathfrak{g}}$-module whose completion $\mathcal{H}_\lambda^\dagger$ will be discussed in section 2.2 below.

In the following we briefly recall the usual treatment of integrable highest weight representation of affine Lie algebras. For the details we refer the reader to Kac's book [19].

A left $\widehat{\mathfrak{g}}^e$ module $\mathcal{M}$ is called a highest weight module with highest weight $\Lambda \in (\mathfrak{h}^e)^*$, if there exists an element $|v\rangle \in \mathcal{M}$ called a *highest weight vector* such that for all $h \in \mathfrak{h}^e$ we have

$$h|v\rangle = \Lambda(h)|v\rangle, \quad \widehat{\mathfrak{n}}_+^e|v\rangle = 0,$$

where

$$\widehat{\mathfrak{n}}_+^e := \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha,$$
$$\Delta_+ := \{\, m\delta + \gamma \mid m > 0, \gamma \in \Delta(\mathfrak{g})_- \cup \{0\} \cup \Delta(\mathfrak{g})_+ \,\}$$

and

$$\mathcal{M} = U(\widehat{\mathfrak{g}}^e)|v\rangle,$$

where $U(\widehat{\mathfrak{g}}^e)$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}^e$. If we put

$$\Lambda := a(\lambda) + \ell\delta$$

and identify the derivation D with the Virasoro operator $-L_0$ defined in the next section (see (2.2.2) below), then the Verma module $\mathcal{M}_\lambda$ and $\mathcal{H}_\lambda$ are the highest weight $\widehat{\mathfrak{g}}^e$-modules. A highest weight $\widehat{\mathfrak{g}}^e$-module $\mathcal{M}$ is called an integrable module, if for any root $\alpha \in \Delta$ of $\mathfrak{g}$ and any integer n, the element $X_\alpha \otimes \xi^n$ acts on $\mathcal{M}$ locally nilpotent, that is, for any element $|\Psi\rangle \in \mathcal{M}$ there exists a positive integer m depending on the element $|\Psi\rangle$ such that $(X_\alpha \otimes \xi^n)^m|\Psi\rangle = 0$. It is known that an irreducible integrable highest weight left $\widehat{\mathfrak{g}}^e$-module is isomorphic to some $\mathcal{H}_\lambda$ by the above identification.

A highest weight integrable $\widehat{\mathfrak{g}}$-module is characterized by the following theorem.

Theorem 2.1.5. *A highest weight $\widehat{\mathfrak{g}}$-module $\mathcal{H}$ is integrable if and only if for any root vector $X_\alpha \in \mathfrak{g}$ and integer n, $X_\alpha(n)$ operates on $\mathcal{H}$ locally nilpotent, that is, for any element $|\varphi\rangle \in \mathcal{H}$ there exists a positive integer m such that*

$$X_\alpha(n)^k|\varphi\rangle = 0, \quad \text{for all } k \geqslant m.$$

2.2. The Segal-Sugawara Form.

In the following we use the following notation freely.

$$X(n) = X \otimes \xi^n, \quad X \in \mathfrak{g},$$
$$X(z) = \sum_{n \in \mathbb{Z}} X(n) z^{-n-1},$$

where z is a variable. Then the normal ordering $\overset{\circ}{\circ}\cdot\cdot\overset{\circ}{\circ}$ is defined by

$$\overset{\circ}{\circ} X(n)Y(m) \overset{\circ}{\circ} = \begin{cases} X(n)Y(m), & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)), & n = m, \\ Y(m)X(n), & n > m. \end{cases}$$

Note that by (2.1.4), if $n > m$ and $X = Y$, we have

$$\overset{\circ}{\circ} X(n)X(m) \overset{\circ}{\circ} = X(n)X(m) - n\delta_{n+m,0}(X, X) \cdot c. \tag{2.2.1}$$

Definition 2.2.1. The *energy-momentum* tensor $T(z)$ of level ℓ is defined by

$$T(z) = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(z) J^a(z) \overset{\circ}{\circ},$$

where $\{J^1, J^2, \dots\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form $(\cdot, \cdot)$ and g^* is the dual Coxeter number of $\mathfrak{g}$. (See (2.2.3) below.)

Put

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(m) J^a(n-m) \overset{\circ}{\circ}. \tag{2.2.2}$$

Then we have the expansion

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

The operator L_n is called the Virasoro operator which acts on $\mathcal{H}_\lambda$. By (2.2.1), if $n \neq 0$, in the definition of L_n, we need not use the normal ordering, that is, we have

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} J^a(m) J^a(n-m).$$

For $n = 0$ we need the normal ordering to define L_0. The operator

$$\frac{1}{2(g^* + \ell)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} J^a(m) J^a(-m)$$

cannot operate on $\mathcal{H}_\lambda$.

The operator L_0 is a generalization of the Casimir element of the simple Lie algebra $\mathfrak{g}$ to the affine Lie algebra $\widehat{\mathfrak{g}}$. Let us briefly discuss the Casimir element Ω of the simple Lie algebra $\mathfrak{g}$. The Casimir element is defined by

$$\Omega = \sum_{a=1}^{\dim \mathfrak{g}} X_a X^a$$

as an element of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, where $\{X_a\}$ is a basis of $\mathfrak{g}$ and $\{X^a\}$ is the dual basis with respect to the Cartan-Killing form. Note that Ω is independent of the choice of a basis. In particular we have

$$\Omega = \sum_{a=1}^{\dim \mathfrak{g}} J^a J^a.$$

Lemma 2.2.2. *Ω belongs to the center of $U(\mathfrak{g})$.*

Proof. It is enough to show that

$$[X, \Omega] = 0$$

for any element of $X \in \mathfrak{g}$. Since $\{J^a\}$ is an orthonormal basis of $\mathfrak{g}$, we have

$$\begin{aligned}
[X, \Omega] &= \sum_{a=1}^{\dim \mathfrak{g}} \{X J^a J^a - J^a J^a X\} \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \{[X, J^a] J^a + J^a [X, J^a]\} \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \left\{ \sum_{b=1}^{\dim \mathfrak{g}} J^b ([X, J^a], J^b) J^a + J^a [X, J^a] \right\} \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \left\{ - \sum_{b=1}^{\dim \mathfrak{g}} J^b (J^a, [X, J^b]) J^a + J^a [X, J^a] \right\} \qquad \text{by (2.1.3)} \\
&= - \sum_{b=1}^{\dim \mathfrak{g}} J^b [X, J^b] + \sum_{a=1}^{\dim \mathfrak{g}} J^a [X, J^a] \\
&= 0.
\end{aligned}$$

This is the desired result. □

Lemma 2.2.3. *Let V_λ be the highest weight left $\mathfrak{g}$-module with the highest weight λ. The Casimir element Ω operates on V_λ as*

$$\{(\lambda, \lambda) + 2(\lambda, \rho)\} \cdot \mathrm{id},$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

Proof. By lemma 2.2.2 and Schur's lemma, Ω acts on V_λ by scalar multiplication. Therefore, it is enough to calculate $\Omega|\lambda\rangle$, where $|\lambda\rangle$ is a highest weight vector of V_λ. Choose non-zero elements $E_\alpha \in \mathfrak{g}_\alpha$, $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$. By (2.1.2) for any element $H \in \mathfrak{h}$ we have

$$\begin{aligned}
([E_\alpha, E_{-\alpha}], H) &= -(E_{-\alpha}, [E_\alpha, H]) \\
&= \alpha(H)(E_\alpha, E_{-\alpha}) \\
&= (E_\alpha, E_{-\alpha})(H_\alpha, H),
\end{aligned}$$

where H_α is the root vector. Hence, we have

$$[E_\alpha, E_{-\alpha}] = (E_\alpha, E_{-\alpha}) H_\alpha.$$

Note that

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$$

if $\alpha + \beta \neq 0$. Let $\{H_1, \ldots, H_\ell\}$ be an orthonormal basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with respect to the Cartan-Killing form. The Casimir element Ω can be expressed in the form

$$\Omega = \sum_{i=1}^{\ell} H_i H_i + \sum_{\alpha \in \Delta} \frac{1}{(E_\alpha, E_{-\alpha})} E_\alpha E_{-\alpha}.$$

As we have

$$H_\lambda = \sum_{i=1}^{\ell} (H_\lambda) H_i = \sum_{i=1}^{\ell} \lambda(H_i) H_i,$$

we have

$$\begin{aligned} \sum_{i=1}^{\ell} H_i H_i |\lambda\rangle &= \sum_{i=1}^{\ell} \lambda(H_i)\lambda(H_i)|\lambda\rangle \\ &= (H_\lambda, H_\lambda)|\lambda\rangle \\ &= (\lambda,\lambda)|\lambda\rangle. \end{aligned}$$

On the other hand we have

$$\sum_{\alpha\in\Delta} \frac{1}{(E_\alpha, E_{-\alpha})} E_\alpha E_{-\alpha} = \sum_{\alpha\in\Delta_+} \frac{1}{(E_\alpha, E_{-\alpha})} \{2 E_\alpha E_{-\alpha} - [E_\alpha, E_{-\alpha}]\}.$$

Since $|\lambda\rangle$ is a highest weight, we have

$$\begin{aligned} E_\alpha E_{-\alpha}|\lambda\rangle &= [E_\alpha, E_{-\alpha}]|\lambda\rangle \\ &= (E_\alpha, E_{-\alpha}) H_\alpha |\lambda\rangle \\ &= (E_\alpha, E_{-\alpha})\lambda(H_\alpha)|\lambda\rangle \\ &= (E_\alpha, E_{-\alpha})(\lambda,\alpha)|\lambda\rangle. \end{aligned}$$

Thus we obtain the desired result. □

The Lie algebra $\mathfrak{g}$ itself is an irreducible left $\mathfrak{g}$-module by the adjoint action

$$\mathrm{ad}(X)(Y) = [X, Y].$$

The highest weight of the adjoint representation is the maximal root θ. Put

$$g^* = 1 + (\theta, \rho). \tag{2.2.3}$$

The number g^* is called the *dual Coxeter number*. For example, for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we have $g^* = n$.

We have the following corollary.

Corollary 2.2.4.

$$\sum_{a=1}^{\dim \mathfrak{g}} \mathrm{ad}\, J^a \,\mathrm{ad}\, J^a(X) = \sum_{a=1}^{\dim \mathfrak{g}} [J^a, [J^a, X]] = 2g^* X.$$

Lemma 2.2.5. *For $X \in \mathfrak{g}$, m, $n \in \mathbb{Z}$, we have*

$$\sum_{a=1}^{\dim \mathfrak{g}} \{[X, J^a](m) J^a(n) + J^a(m)[X, J^a](n)\} = 0.$$

Proof.

$$\begin{aligned}\sum_{a=1}^{\dim \mathfrak{g}} [X, J^a](m)J^a(n) &= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{b=1}^{\dim \mathfrak{g}} ([X, J^a], J^b)J^b(m)J^a(n) \\ &= -\sum_{a=1}^{\dim \mathfrak{g}} \sum_{b=1}^{\dim \mathfrak{g}} (J^a, [X, J^b])J^b(m)J^a(n) \quad \text{by (2.1.2)} \\ &= -\sum_{b=1}^{\dim \mathfrak{g}} \sum_{a=1}^{\dim \mathfrak{g}} J^b(m)(J^a, [X, J^b])J^a(n) \\ &= -\sum_{b=1}^{\dim \mathfrak{g}} J^b(m)[X, J^b](n) \\ &= -\sum_{a=1}^{\dim \mathfrak{g}} J^a(m)[X, J^a](n)\end{aligned}$$

This proves the lemma. □

Lemma 2.2.6. *The set $\{L_n\}$ forms a Virasoro algebra and we have*

$$\begin{aligned}[L_n, X(m)] &= -mX(n+m), \quad \text{for } X \in \mathfrak{g}, \\ [L_n, L_m] &= (n-m)L_{n+m} + \frac{c_v}{12}(n^3 - n)\delta_{n+m,0},\end{aligned}$$

where

$$c_v = \frac{\ell \dim \mathfrak{g}}{g^* + \ell}$$

is the central charge of the Virasoro algebra.

Proof. By (2.2.2) we have

$$\begin{aligned}(2g^* + \ell)[X(m), L_n] &= \Big[X(m), \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \in \mathbb{Z}} {}^{\circ}_{\circ} J^a(-j)J^a(n+j){}^{\circ}_{\circ}\Big] \\ &= \sum_{a=1}^{\dim \mathfrak{g}} \Big\{ \Big[X(m), \sum_{j \geqslant -\frac{n}{2}} J^a(-j)J^a(n+j)\Big] \\ &\quad + \Big[X(m), \sum_{j < -\frac{n}{2}} J^a(n+j)J^a(-j)\Big]\Big\}. \qquad (2.2.4)\end{aligned}$$

The first term of the right hand side of (2.2.4) is rewritten in the following form.

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \Big[X(m), \sum_{j\geqslant -\frac{n}{2}} J^a(-j)J^a(n+j)\Big] \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j\geqslant -\frac{n}{2}} \{[X(m), J^a(-j)]J^a(n+j) + J^a(-j)[X(m), J^a(n+j)]\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j\geqslant -\frac{n}{2}} \{[X, J^a](m-j)J^a(n+j) + \ell(X, J^a)\delta_{m-j,0}J^a(n+j) \\
&\qquad + J^a(-j)[X, J^a](m+n+j) + \ell(X, J^a)\delta_{m+n+j,0}J^a(-j)\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j\geqslant -\frac{n}{2}} \{[X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j)\} \\
&\qquad + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j\geqslant -\frac{n}{2}} \{m\ell(X, J^a)\delta_{m+n+j,0}J^a(-j) + m\ell(X, J^a)\delta_{m-j,0}J^a(n+j)\}\,.
\end{aligned}$$

Hence, for any integer n we obtain the following formula

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \Big\{\Big[X(m), \sum_{j\geqslant -\frac{n}{2}} J^a(-j)J^a(n+j)\Big]\Big\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>-\frac{n}{2}} \{[X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j)\} \\
&\qquad + m\ell\delta_{m-j,0}X(n+j) + \sum_{j\geqslant -\frac{n}{2}} \{m\ell\delta_{m+n+j,0}X(-j) + m\ell\delta_{m-j,0}X(n+j)\}.
\end{aligned}$$

The second term of the equality (2.2.4) is rewritten the following way

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \Big[X(m), \sum_{j<-\frac{n}{2}} J^a(n+j)J^a(-j)\Big] \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j<-\frac{n}{2}} \{[X(m), J^a(n+j)])J^a(-j) + J^a(n+j)[X(m), J^a(-j)]\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j<-\frac{n}{2}} \{[X, J^a](m+n+j)J^a(-j) + J^a(n+j)[X, J^a](m-j) \\
&\qquad + m\ell(X, J^a)\delta_{m+n+j,0}J^a(-j) + m\ell(X, J^a)\delta_{m-j,0}J^a(n+j)\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j<-\frac{n}{2}} \{[X, J^a](m+n+j)J^a(-j) + J^a(n+j)[X, J^a](m-j)\} \\
&\qquad + \sum_{j<-\frac{n}{2}} \{m\ell\delta_{m+n+j,0}X(-j) + m\ell(X, J^a)\delta_{m-j,0}X(n+j)\}\} \\
&\quad = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>-\frac{n}{2}} \{[X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j)\} \\
&\qquad + \sum_{j<-\frac{n}{2}} \{m\ell\delta_{m+n+j,0}X(-j) + m\ell\delta_{m-j,0}X(n+j).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&2(g^*+\ell)[X(m),L_n]\\
&\quad=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j>-\frac{n}{2}}\{[X,J^a](m-j)J^a(n+j)+J^a(-j)[X,J^a](m+n+j)\}\\
&\qquad+\sum_{j\in\mathbb{Z}}\{m\ell\delta_{m+n+j,0}X(-j)+m\ell\delta_{m-j,0}X(n+j)\}.
\end{aligned}$$

Now

$$\sum_{j\in\mathbb{Z}}\{m\ell\delta_{m+n+j,0}X(-j)+m\ell\delta_{m-j,0}X(n+j)\}=2m\ell X(n+m).$$

Note that by (2.1.3) we have

$$([X,J^a],J^a)=0. \tag{2.2.5}$$

Let us consider the case in which m in non-negative. Then, we have

$$\begin{aligned}
&2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j>-\frac{n}{2}}\{[X,J^a](m-j)J^a(n+j)+J^a(-j)[X,J^a](m+n+j)\}\\
&\quad=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m-\frac{n}{2}\geqslant j>-\frac{n}{2}}[X,J^a](m-j)J^a(n+j)\\
&\quad=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m-\frac{n}{2}\geqslant j>-\frac{n}{2}}\{J^a(n+j)[X,J^a](m-j)+[[X,J^a],J^a](m+n)\}\\
&\quad=\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m-\frac{n}{2}\geqslant j>-\frac{n}{2}}\{[X,J^a](m-j)J^a(n+j)+J^a(n+j)[X,J^a](m-j)\}\\
&\qquad+m\sum_{a=1}^{\dim\mathfrak{g}}[[X,J^a],J^a](m+n)\\
&\quad=2mg^*X(m+n)\qquad\text{by Corollary 2.2.4 and Lemma 2.2.5 .}
\end{aligned}$$

Thus we obtain

$$2(g^*+\ell)[X(m),L_n]=2m(g^*+\ell)X(m+n).$$

For negative m a similar argument proves the equality. Now the second equality is an easy consequence of the first one and we leave the proof to the reader. □

Corollary 2.2.7.

$$[L_n,X(z)]=z^n\left(z\frac{d}{dz}+n+1\right)X(z).$$

For $X\in\mathfrak{g}$, $f=f(z)\in\mathbb{C}((z))$ and $\underline{\ell}=\ell(z)\frac{d}{dz}\in\mathbb{C}((z))\frac{d}{dz}$ we use the following notation.

$$\begin{aligned}
X[f]&=\operatorname*{Res}_{z=0}(X(z)f(z)dz),\\
T[\underline{\ell}]&=\operatorname*{Res}_{z=0}(T(z)\ell(z)dz).
\end{aligned}$$

In particular, we have

$$L_0=T\left[\xi\frac{d}{d\xi}\right]. \tag{2.2.6}$$

Thus, by identifying the derivation D with L_0, $\widehat{\mathfrak{g}}^e$ acts on $\mathcal{H}_\lambda$. (See section 2.1.)

Lemma 2.2.8. *$X[f]$ and $T[\underline{\ell}]$ act on $\mathcal{H}_\lambda$ and we have*

$$\begin{aligned} X[f] &= X \otimes f(\xi), \\ [T[\underline{\ell}], X[f]] &= -X[\underline{\ell}(f)], \\ [T[\underline{\ell}], T[\underline{m}]] &= -T[[\underline{\ell}, \underline{m}]] + \frac{c_v}{12} \operatorname*{Res}_{z=0}(\ell''' m dz). \end{aligned} \tag{2.2.7}$$

Proof. Here, we give a proof of the third formula. Put

$$\begin{aligned} \underline{\ell} &= \ell(\xi)\frac{d}{d\xi}, \qquad \ell(\xi) = \sum_{n=-n_0}^{\infty} \ell_n \xi^{n+1}, \\ \underline{m} &= m(\xi)\frac{d}{d\xi}, \quad m(\xi) = \sum_{n=-n_0}^{\infty} m_n \xi^{n+1}. \end{aligned}$$

Then we have

$$\begin{aligned} [T[\underline{\ell}], T[\underline{m}]] &= \sum_{k,j} [\ell_j L_j, m_k L_k] \\ &= \sum_{k,j} \ell_j m_k [L_j, L_k] \\ &= \sum_{k,j} \ell_j m_k \left\{ (j-k) L_{j+k} + \frac{c_v}{12}(j^3 - j)\delta_{j+k,0} \right\}. \end{aligned}$$

On the other hand, we have

$$[\underline{\ell}, \underline{m}] = \sum_n \Big(\sum_{j+k=n} \ell_j m_k (k-j) \Big) \xi^{n+1} \frac{d}{d\xi}.$$

Moreover we have

$$\sum_{k,j} \ell_j m_k (j^3 - j)\delta_{j+k,0} = \sum_j \ell_j m_{-j}(j^3 - j) = \operatorname*{Res}_{z=0}(\ell'''(z) m(z) dz).$$

This proves the third formula. □

Remark 2.2.9. In the last formula of (2.2.7) we can use an other expression based on the following equalities.

$$\operatorname*{Res}_{z=0}(\ell'''(z) m(z) dz) = -\operatorname*{Res}_{z=0}(m'''(z)\ell(z) dz) = \frac{1}{2} \operatorname*{Res}_{z=0} \left(\begin{vmatrix} \ell'(z) & m'(z) \\ \ell''(z) & m''(z) \end{vmatrix} dz \right).$$

To define a filtration $\{F_\bullet\}$ on $\mathcal{H}_\lambda$, we first define the subspace $\mathcal{H}_\lambda(d)$ of $\mathcal{H}_\lambda$ for a non-negative integer d by

$$\mathcal{H}_\lambda(d) = \{\, |v\rangle \in \mathcal{H}_\lambda \mid L_0|v\rangle = (d + \Delta_\lambda)|v\rangle \,\}, \tag{2.2.8}$$

where

$$\Delta_\lambda = \frac{(\lambda, \lambda) + 2(\lambda, \rho)}{2(g^* + \ell)}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

Note that by (2.2.2), on V_λ the operator L_0 acts as

$$L_0|v\rangle = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} J^a J^a |v\rangle,$$

for $|v\rangle \in \mathcal{H}_\lambda$. Hence by Lemma 2.2.3 L_0 acts on V_λ by scalar multiplication by Δ_λ. For a positive integer m and $|v\rangle \in V_\lambda$, we have

$$\begin{aligned} L_0 X(-m)|v\rangle &= L_0 X(-m)|v\rangle + mX(-m)|v\rangle \\ &= (\Delta_\lambda + m)X(-m)|v\rangle. \end{aligned}$$

Hence, we have $X(-m)|v\rangle \in F_m\mathcal{H}_\lambda$. Similarly, for positive integers $m_1, \ldots, m_k$, we have

$$\begin{aligned} &L_0 X_1(m_1)\cdots X_k(-m_k)|v\rangle \\ &\qquad = (\Delta_\lambda + m_1 + \cdots + m_k)X_1(m_1)\cdots X_k(-m_k)|v\rangle, \quad |v\rangle \in V_\lambda. \end{aligned}$$

¿From this, it is easy to show that $\mathcal{H}_\lambda(d)$ is a finite dimensional vector space and we have

$$\mathcal{H}_\lambda = \bigoplus_{d=0}^{\infty} \mathcal{H}_\lambda(d).$$

For a negative integer $-d$ we put

$$\mathcal{H}_\lambda(-d) = \{0\}.$$

Now we define the filtration $\{F_p\mathcal{H}_\lambda\}$, where

$$F_p\mathcal{H}_\lambda = \sum_{d=0}^{p} \mathcal{H}_\lambda(d). \tag{2.2.9}$$

Note that this is an increasing filtration.

Put

$$\mathcal{H}_\lambda^\dagger(d) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda(d), \mathbb{C}). \tag{2.2.10}$$

Then the dual space $\mathcal{H}_\lambda^\dagger$ of $\mathcal{H}_\lambda$ is defined to be

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda, \mathbb{C}) = \prod_{d=0}^{\infty} \mathcal{H}_\lambda^\dagger(d). \tag{2.2.11}$$

By our definition $\mathcal{H}_\lambda^\dagger$ is a right $\widehat{\mathfrak{g}}$-module. A decreasing filtration $\{F^p\mathcal{H}_\lambda^\dagger\}$ is defined by

$$F^p\mathcal{H}_\lambda^\dagger = \prod_{d \geqslant p} \mathcal{H}_\lambda^\dagger(d). \tag{2.2.12}$$

There is a canonical perfect bilinear pairing

$$\langle \cdot | \cdot \rangle \colon \mathcal{H}_\lambda^\dagger \times \mathcal{H}_\lambda \longrightarrow \mathbb{C}, \tag{2.2.13}$$

which satisfies the following equality for all $a \in \widehat{\mathfrak{g}}$.

$$\langle u|av\rangle = \langle ua|v\rangle, \quad \text{for all } \langle u| \in \mathcal{H}_\lambda^\dagger \text{and } |v\rangle \in \mathcal{H}_\lambda.$$

Note that the filtrations $\{F_p\}$ and $\{F^p\}$ define a topology on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$, respectively. With respect to this topology $\mathcal{H}_\lambda^\dagger$ is complete and is the completion of the integrable highest weight right $\widehat{\mathfrak{g}}$-module with the lowest weight λ. Put

$$V_\lambda^\dagger = \{\, \langle v| \in \mathcal{H}_\lambda^\dagger \mid \langle v|\widehat{\mathfrak{g}}_- = 0 \,\}.$$

It is easy to show that $V_\lambda^\dagger = \mathcal{H}_\lambda^\dagger(0)$ and $V_\lambda^\dagger$ is the irreducible right $\mathfrak{g}$-module with lowest weight λ. The integrable highest weight right $\widehat{\mathfrak{g}}$-module with lowest weight λ is generated by $V_\lambda^\dagger$ over $\widehat{\mathfrak{g}}_+$ with only one relation

$$\langle \lambda|(X_{-\theta} \otimes \xi)^{\ell-(\theta,\lambda)+1} = 0.$$

Lemma 2.2.10.

$$\begin{aligned} X(m)\mathcal{H}_\lambda(d) &\subset \mathcal{H}_\lambda(d-m), \\ L_m\mathcal{H}_\lambda(d) &\subset \mathcal{H}_\lambda(d-m), \\ \mathcal{H}_\lambda^\dagger(d)X(m) &\subset \mathcal{H}_\lambda^\dagger(d+m), \\ \mathcal{H}_\lambda^\dagger(d)L_m &\subset \mathcal{H}_\lambda^\dagger(d+m). \end{aligned}$$

In the similar manner we can introduce a filtration on the Verma module $\mathcal{M}_\lambda$.

Proposition 2.2.11. *For a root vector $X_\alpha \in \mathfrak{g}_\alpha$ of the simple Lie algebra $\mathfrak{g}$ and any element $f(\xi) \in \mathbb{C}((\xi))$ the actions of $X_\alpha \otimes f(\xi)$ on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$ are locally nilpotent.*

Proof. We know that $X_\alpha(n)$ operates locally nilpotent on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$. By Theorem 3.2.4 (1) below it is easy to show that $X_\alpha \otimes f(\xi)$ also operates locally nilpotent. □

Now let us introduce the left $\mathfrak{g}$-module structure on $\mathcal{H}_\lambda^\dagger$ by

$$X(n)\langle\Phi| := -\langle\Phi|X(-n).$$

It is easy to check that this indeed defines a left $\mathfrak{g}$-module structure on $\mathcal{H}_\lambda^\dagger$.

Now we give the relationship between the *left* $\mathfrak{g}$-modules $\mathcal{H}_\lambda^\dagger$ and $\mathcal{H}_{\lambda^\dagger}$. For an element $\lambda \in P_\ell$ put

$$\lambda^\dagger = -w(\lambda),$$

where w is the longest element of the Weyl group of the simple Lie algebra $\mathfrak{g}$ (in other word, $w(\Delta_+) = \Delta_-$). Note that $\lambda^\dagger$ is also characterized by the fact that $-\lambda^\dagger$ is the lowest weight of the $\mathfrak{g}$-module V_λ. For example, for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, we have $\lambda^\dagger = \lambda$.

Lemma 2.2.12. *There exists a bilinear pairing*

$$(\cdot|\cdot)\colon \mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger} \longrightarrow \mathbb{C},$$

unique up to a constant multiple, such that we have

$$(X(n)u|v) + (u|X(-n)v) = 0,$$

for any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $|u\rangle \in \mathcal{H}_\lambda$, $|v\rangle \in \mathcal{H}_{\lambda^\dagger}$. Moreover $(\cdot|\cdot)$ is zero on $\mathcal{H}_\lambda(d) \times \mathcal{H}_{\lambda^\dagger}(d')$, if $d \neq d'$.

Proof. Since $V_\lambda \otimes V_{\lambda^\dagger}$, considered as a $\mathfrak{g}$-module by the diagonal action, contains only the one-dimensional trivial $\mathfrak{g}$-module $\mathbb{C}|0_{\lambda,\lambda^\dagger}\rangle$ (for the properties of the vector $|0_{\lambda,\lambda^\dagger}\rangle$ see the proof of Theorem 3.3.5 below), we have a bilinear form $(\cdot|\cdot) \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\lambda^\dagger}, \mathbb{C})$ unique up to multiplication by a constant. Assume that we have a bilinear form $(\cdot|\cdot) \in \mathrm{Hom}(F_p\mathcal{H}_\lambda \otimes F_p\mathcal{H}_{\lambda^\dagger}, \mathbb{C})$ with the desired properties. For an element

$$X(-m)|u\rangle \in F_{p+1}\mathcal{H}_\lambda, \quad |u\rangle \in F_p\mathcal{H}_\lambda, \quad m > 0$$

and an element $|v\rangle \in F_{p+1}\mathcal{H}_{\lambda^\dagger}$ define

$$(X(-m)u|v) = -(u|X(m)v). \tag{2.2.14}$$

Note that since $X(m)|v\rangle \in F_{p+1-m}\mathcal{H}_{\lambda^\dagger}$, the right hand side is defined already. It is easy to show that in this way we can define the bilinear form $(\cdot|\cdot)$ satisfying the conditions of Lemma 2.2.12. □

Corollary 2.2.13. *There is a canonical left $\mathfrak{g}$-module isomorphism*

$$\mathcal{H}_\lambda^\dagger \simeq \widehat{\mathcal{H}}_{\lambda^\dagger}$$

where $\widehat{\mathcal{H}}_{\lambda^\dagger}$ is the completion of $\mathcal{H}_{\lambda^\dagger}$ with respect to the filtration $\{F_p\}$.

Note that Garland [15] has introduced a natural hermitian form on $\mathcal{H}_\lambda$.

3. The Space of Conformal Blocks and Correlation Functions

3.1. The Space of Conformal Blocks.

In this section, to an N-pointed stable curve with formal neighbourhoods $\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N; \eta_1, \eta_2, \dots, \eta_N)$ we associate the space of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ and its dual space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$, where $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$, $\lambda_j \in P_\ell$. For that purpose we first define a *generalized affine Lie algebra* $\widehat{\mathfrak{g}}_N$.

Definition 3.1.1. Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers. A Lie algebra $\widehat{\mathfrak{g}}_N$ is defined as

$$\widehat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi_j)) \oplus \mathbb{C}c$$

with the following commutation relations.

$$[(X_j \otimes f_j), (Y_j \otimes g_j)] = ([X_j, Y_j] \otimes f_j g_j) + c \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0} (g_j df_j), \tag{3.1.1}$$

where (a_j) means $(a_1, a_2, \dots, a_N)$ and c belongs to the center of $\widehat{\mathfrak{g}}_N$.

We also put

$$\widehat{\mathfrak{g}}(\mathfrak{X}) = \mathfrak{g} \otimes_{\mathbb{C}} H^0\left(C, \mathcal{O}_C\left(* \sum_{j=1}^{N} Q_j\right)\right). \tag{3.1.2}$$

By Lemma 1.1.5 we have a natural embedding

$$t\colon H^0\left(C, \mathcal{O}_C\left(* \sum_{j=1}^{N} Q_j\right)\right) \longhookrightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j)).$$

In the following we shall often regard $H^0\left(C, \mathcal{O}_C\left(* \sum_{j=1}^{N} Q_j\right)\right)$ as a subspace of the direct sum $\bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))$. By Lemma 1.1.15 we have the following lemma.

Lemma 3.1.2. *$\widehat{\mathfrak{g}}(\mathfrak{X})$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$.*

Let us fix a non-negative integer ℓ. For each $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, a left $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}_{\vec{\lambda}}$ and a right $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}^{\dagger}_{\vec{\lambda}}$ are defined by

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_N},$$
$$\mathcal{H}^{\dagger}_{\vec{\lambda}} = \mathcal{H}^{\dagger}_{\lambda_1} \widehat{\otimes}_{\mathbb{C}} \cdots \widehat{\otimes}_{\mathbb{C}} \mathcal{H}^{\dagger}_{\lambda_N}.$$

For each element $X_j \in \mathfrak{g}$, $f(\xi_j) \in \mathbb{C}((\xi_j))$, the action ρ_j of $X_j[f_j]$ on $\mathcal{H}_{\vec{\lambda}}$ is given by

$$\rho_j(X_j[f_j])|v_1 \otimes \cdots \otimes v_N\rangle = |v_1 \otimes \cdots \otimes v_{j-1} \otimes (X_j[f_j])v_j \otimes v_{j+1} \otimes \cdots \otimes v_N\rangle, \tag{3.1.3}$$

where $|v_1 \otimes \cdots \otimes v_N\rangle$ means $|v_1\rangle \otimes \cdots \otimes |v_N\rangle$, $|v_j\rangle \in \mathcal{H}_{\lambda_j}$. The left $\widehat{\mathfrak{g}}_N$-action is given by

$$(X_1 \otimes f_1, \dots, X_N \otimes f_N)|v_1 \otimes \cdots \otimes v_N\rangle = \sum_{j=1}^{N} \rho_j(X_j[f_j])|v_1 \otimes \cdots \otimes v_N\rangle. \tag{3.1.4}$$

Similarly, the right $\widehat{\mathfrak{g}}_N$-action on $\mathcal{H}^{\dagger}_{\vec{\lambda}}$ is defined by

$$\langle u_1 \otimes \cdots u_N|(X_1 \otimes f_1, \dots, X_N \otimes f_N) = \sum_{j=1}^{N} \langle u_1 \otimes \cdots \otimes u_N|\rho_j(X_j[f_j]). \tag{3.1.5}$$

As a Lie subalgebra, $\widehat{\mathfrak{g}}(\mathfrak{X})$ operates on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}_{\vec{\lambda}}^{\dagger}$ as

$$(X \otimes f)|v_1 \otimes \cdots \otimes v_N\rangle = \sum_{j=1}^{N} \rho_j(X \otimes t_j(f))|v_1 \otimes \cdots \otimes v_N\rangle, \tag{3.1.6}$$

$$\langle u_1 \otimes \cdots \otimes u_N|(X \otimes f) = \sum_{j=1}^{N} \langle u_1 \otimes \cdots \otimes u_N|\rho_j(X \otimes t_j(f)). \tag{3.1.7}$$

The pairing $\langle \cdot | \cdot \rangle$ introduced in (2.2.13) induces a perfect bilinear pairing

$$\begin{gathered} \langle \cdot | \cdot \rangle \colon \mathcal{H}_{\vec{\lambda}}^{\dagger} \times \mathcal{H}_{\vec{\lambda}} \longrightarrow \mathbb{C} \\ (\langle u_1 \otimes \cdots \otimes u_N|, |v_1 \otimes \cdots \otimes u_N\rangle) \longmapsto \langle u_1|v_1\rangle\langle u_2|v_2\rangle \cdots \langle u_N|v_N\rangle, \end{gathered} \tag{3.1.8}$$

which is $\widehat{\mathfrak{g}}_N$-invariant:

$$\langle \Psi(X_j \otimes f_j)|\Phi\rangle = \langle \Psi|(X_j \otimes f_j)\Phi\rangle.$$

Now we are ready to define the space of vacua attached to $\mathfrak{X}$.

Definition 3.1.3. Assume that $\mathfrak{X}$ enjoys the property (Q) in section 1.1. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \mathcal{H}_{\vec{\lambda}}/\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}. \tag{3.1.9}$$

The vector space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is called the *space of covacua* attached to $\mathfrak{X}$. The *space of vacua* attached to $\mathfrak{X}$ is defined as

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}), \mathbb{C}). \tag{3.1.10}$$

In case $\mathfrak{X}$ does not satisfy the property (Q), we use Proposition 3.3.1 below to define the space of vacua. The space of vacua is often called a *conformal block* by physicists.

From the definition (3.1.10) the following lemma follows easily.

Lemma 3.1.4.

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \{\, \langle \Psi| \in \mathcal{H}_{\vec{\lambda}}^{\dagger} \mid \langle \Psi|\widehat{\mathfrak{g}}(\mathfrak{X}) = 0 \,\}. \tag{3.1.11}$$

Moreover, the pairing (3.1.8) induces a perfect pairing

$$\langle \cdot | \cdot \rangle \colon \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow \mathbb{C}. \tag{3.1.12}$$

By our definition it is not clear whether $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ is a finite-dimensional vector space. Now we prove the following important fact.

Theorem 3.1.5. *$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ are finite-dimensional vector spaces.*

In Chapter 4 we shall prove that $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ depends only on the genus $g(C)$ of the curve C and $\vec{\lambda}$.

To prove Theorem 3.1.5, first we shall prove the following proposition.

Proposition 3.1.6. *For each non-negative integer m, the vector space*

$$V_m = \mathcal{H}_{\vec{\lambda}} \Big/ \Big(\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{j=1}^{N} \mathbb{C}[\xi_j^{-1}]\xi_j^{-m}\Big)\mathcal{H}_{\vec{\lambda}}$$

is finite-dimensional.

Note that $\mathfrak{g} \otimes_{\mathbb{C}} (\bigoplus_{j=1}^{N} \mathbb{C}[\xi_j]\xi_j^{-m})$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$.

Proof of Proposition 3.1.6. We introduce a filtration $\{F_\bullet\}$ of $\mathcal{H}_{\vec{\lambda}}$ by

$$F_p\mathcal{H}_{\vec{\lambda}} = \sum_{d \leqslant p} \mathcal{H}_{\vec{\lambda}}(d), \tag{3.1.14}$$

where we put

$$\mathcal{H}_{\vec{\lambda}}(d) = \sum_{d_1+\cdots+d_N=d} \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N). \tag{3.1.15}$$

Note that

$$F_0\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\vec{\lambda}}(0) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}.$$

We denote this vector space by $V_{\vec{\lambda}}$. By Theorem 2.1.4 $\mathcal{H}_{\vec{\lambda}}$ is generated by $V_{\vec{\lambda}}$ as a $U\big(\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{j=1}^N \mathbb{C}[\xi_j^{-1}]\big)$-module, where $U(\mathfrak{q})$ denotes the enveloping algebra of a Lie algebra $\mathfrak{q}$. Let $\overline{V}$ denote the image of $V_{\vec{\lambda}}$ in V_m. Put

$$\overline{\mathfrak{g}} = \mathfrak{g} \otimes \Big(\bigoplus_{j=1}^N (\mathbb{C}[\xi_j^{-1}]/(\xi_j^{-m}))\Big).$$

Then $\overline{V}$ is a finite-dimensional vector space and we have

$$V_m = U(\overline{\mathfrak{g}}) \cdot \overline{V}.$$

Hence, V_m is a finite $U(\overline{\mathfrak{g}})$-module. Now we define filtrations $\{G_\bullet\}$ on $U(\overline{\mathfrak{g}})$ and V_m as follows.

$$G_nU(\overline{\mathfrak{g}}) = \begin{cases} 0, & n < 0 \\ \mathbb{C} \cdot 1, & n = 0 \\ G_{n-1}U(\overline{\mathfrak{g}}) + \overline{\mathfrak{g}}G_{n-1}U(\overline{\mathfrak{g}}), & n \geqslant 1, \end{cases}$$
$$G_nV_m = G_nU(\overline{\mathfrak{g}}) \cdot \overline{V}.$$

Then we have

$$G_kU(\overline{\mathfrak{g}})G_nU(\overline{\mathfrak{g}}) \subset G_{k+n}U(\overline{\mathfrak{g}}),$$
$$[G_kU(\overline{\mathfrak{g}}), G_nU(\overline{\mathfrak{g}})] \subset G_{k+n-1}U(\overline{\mathfrak{g}}),$$
$$G_kU(\overline{\mathfrak{g}})G_mV_m \subset G_{k+n}V_m.$$

Now let us consider the associated graded objects $Gr_\bullet^G U(\overline{\mathfrak{g}})$ and $Gr_\bullet^G V_m$. By the Poincaré-Birkhoff-Witt theorem, $Gr_\bullet^G U(\overline{\mathfrak{g}})$ is the symmetric algebra $S^*(\overline{\mathfrak{g}})$ over $\overline{\mathfrak{g}}$, hence a Noetherian commutative ring. Moreover, $Gr_\bullet^G U(\overline{\mathfrak{g}})$ has a Poisson bracket $\{\cdot,\cdot\}$ defined by

$$\{\overline{P}, \overline{Q}\} = \overline{[P, Q]},$$

where

$$P \in G_kU(\overline{\mathfrak{g}}), \quad Q \in G_mU(\overline{\mathfrak{g}})$$

and

$$\overline{P} \in G_kU(\overline{\mathfrak{g}})/G_{k-1}U(\overline{\mathfrak{g}}), \quad \overline{Q} \in G_nkU(\overline{\mathfrak{g}})/G_{n-1}U(\overline{\mathfrak{g}}).$$

Hence, if $a, b \in \overline{\mathfrak{g}} \subset G_1U(\overline{\mathfrak{g}})$, then

$$\{a, b\} = [a, b].$$

Let us consider the annihilation ideal

$$\mathfrak{a} = \{\, a \in Gr_\bullet^G U(\overline{\mathfrak{g}}) \mid a \cdot Gr_\bullet^G V_m = 0 \,\}.$$

Gabber's theorem [14] says that the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ is closed under the Poisson bracket. By the integrability of our $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\vec{\lambda}}$, for each element $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$ and $n \in \mathbb{Z}$, $X_\alpha \otimes \xi_j^{-n}$ operates locally nilpotent on $\mathcal{H}_{\vec{\lambda}}$ (see Theorem 2.1.5), hence we have

$$\overline{X_\alpha \otimes \xi_j^{-n}} \in \sqrt{\mathfrak{a}},$$

for $n = 0, 1, 2, \ldots, m-1$. On the other hand, by Gabber's theorem we have

$$\overline{H_\alpha \otimes \xi_j^{-n}} = \{\overline{X_\alpha \otimes 1}, \overline{X_{-\alpha} \otimes \xi_j^{-n}}\} \in \sqrt{\mathfrak{a}},$$

for $n = 0, 1, \ldots, m-1$. This means that $\overline{\mathfrak{g}} \subset \sqrt{\mathfrak{a}}$. Hence, $\sqrt{\mathfrak{a}}$ contains a maximal ideal $\overline{\mathfrak{g}} Gr_\bullet^G U(\overline{\mathfrak{g}})$. Hence, $Gr_\bullet^G U(\overline{\mathfrak{g}})/\mathfrak{a}$ is an Artinian ring. Since $Gr_\bullet^G V_m$ is a finite $Gr_\bullet^G U(\overline{\mathfrak{g}})/\mathfrak{a}$-module, $Gr_\bullet^G V_m$ is a finite-dimensional vector space. Hence, V_m is a finite-dimensional vector space. This proves Proposition 3.1.6. $\square$

To prove Theorem 3.1.5 we need to introduce filtrations $\{F_\bullet\}$ on $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ and $\widehat{\mathfrak{g}}_N$. First we introduce a filtration $\{F_\bullet\}$ on $\mathbb{C}((\xi_j))$ by

$$F_p\mathbb{C}((\xi_j)) = \mathbb{C}[[\xi_j]]\xi_j^{-p}.$$

The filtration $\{F_\bullet\}$ on $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ is defined by

$$F_p \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) = \bigoplus_{j=1}^N F_p\mathbb{C}((\xi_j)).$$

The filtration $\{F_\bullet\}$ on $\widehat{\mathfrak{g}}_N$ is defined by

$$F_p\widehat{\mathfrak{g}}_N = \begin{cases} \mathfrak{g} \otimes_{\mathbb{C}} (\bigoplus_{j=1}^N F_p\mathbb{C}((\xi_j))) \oplus \mathbb{C} \cdot c, & p \geqslant 0 \\ \mathfrak{g} \otimes_{\mathbb{C}} (\bigoplus_{j=1}^N F_p\mathbb{C}((\xi_j))), & p < 0. \end{cases} \tag{3.1.16}$$

Note that the associated graded objects have the form

$$Gr_\bullet^F \Big(\bigoplus_{j=1}^N \mathbb{C}((\xi_j))\Big) = \bigoplus_{j=1}^N \mathbb{C}[\xi_j, \xi_j^{-1}],$$

$$Gr_\bullet(\widehat{\mathfrak{g}}_N) = \mathfrak{g} \otimes_{\mathbb{C}} \Big(\bigoplus_{j=1}^N \mathbb{C}[\xi_j, \xi_j^{-1}]\Big) \oplus \mathbb{C} \cdot c.$$

By (3.1.14), (3.1.15), (3.1.16) and Lemma 2.2.4 we have

$$F_p\widehat{\mathfrak{g}}_N \cdot F_q\mathcal{H}_{\vec{\lambda}} \subset F_{p+q}\mathcal{H}_{\vec{\lambda}}. \tag{3.1.17}$$

Now the filtrations $\{F_\bullet\}$ on $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ and on $\widehat{\mathfrak{g}}_N$ induce filtrations on the vector spaces $H^0\big(C, \mathcal{O}_C\big(*\sum_{j=1}^N Q_j\big)\big)$ and $\widehat{\mathfrak{g}}(\mathfrak{X})$. (We always regard

$$H^0\Big(C, \mathcal{O}_C\Big(*\sum_{j=1}^N Q_j\Big)\Big)$$

as a subspace of $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ by the embedding t given in (1.1.6).) The following lemma plays an important role in proving Theorem 3.1.5.

Lemma 3.1.7. *If $m > \frac{1}{N}(2g(C)-1)+1$, then we have a natural injection*

$$\bigoplus_{j=1}^{N} \mathbb{C}[\xi_j^{-1}]\xi_j^{-m} \hookrightarrow Gr_\bullet^F H^0\Big(C, \mathcal{O}_C\Big(*\sum_{j=1}^{N} Q_j\Big)\Big).$$

Proof. We have

$$\dim_{\mathbb{C}} H^0\Big(C, \mathcal{O}_C\Big(\sum_{j=1}^{N} n_j Q_j\Big)\Big) = \sum_{j=1}^{N} n_j + 1 - g(C)$$

if $\sum_{j=1}^{N} n_j \geqslant 2g(C)-1$. Hence, if $k > (2g(C)-1)/N$, then we get that

$$\dim_{\mathbb{C}} H^0\Big(C, \mathcal{O}_C\Big((k-1)\sum_{j=1}^{N} Q_j\Big)\Big) = N(k-1)+1-g,$$

$$\dim_{\mathbb{C}} H^0\Big(C, \mathcal{O}_C\Big((k-1)\sum_{j\neq i}^{N} Q_j + kQ_i\Big)\Big) = N(k-1)+2-g.$$

Therefore, there is an element $f \in H^0\big(C, \mathcal{O}_C\big(*\sum_{j=1}^{N} Q_j\big)\big)$ which has a pole of order k at Q_i and a pole of order $\leqslant k-1$ at Q_j, $j \neq i$. This proves the lemma. □

Corollary 3.1.8. *For any positive integer $m \geqslant \frac{1}{N}(2g(C)-1)+1$, there is a natural injection*

$$\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{j=1}^{N} \mathbb{C}[\xi_j]\xi_j^{-m} \hookrightarrow Gr_\bullet^F \widehat{\mathfrak{g}}(\mathfrak{X}).$$

To prove Theorem 3.1.5, we need one more lemma.

Lemma 3.1.9.

$$F_p(\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}) = \sum_{p_1+p_2=p} F_{p_1}\widehat{\mathfrak{g}}(\mathfrak{X})F_{p_2}\mathcal{H}_{\vec{\lambda}}.$$

Proof. Note that in the above equality the right hand side is contained in the left hand side. Therefore, assume

$$F_p(\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}) \supsetneqq \sum_{p_1+p_2=p} F_{p_1}\widehat{\mathfrak{g}}(\mathfrak{X})F_{p_2}\mathcal{H}_{\vec{\lambda}}.$$

Then, there is an element

$$(X \otimes f)|\Phi\rangle \in F_p(\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}})$$

such that

$$X \otimes f \in F_q(\widehat{\mathfrak{g}}(\mathfrak{X})) \setminus F_{q-1}(\widehat{\mathfrak{g}}(\mathfrak{X})),$$
$$|\Phi\rangle \in F_r\mathcal{H}_{\vec{\lambda}}$$

for some q and r with $q+r > p$. For each j there is an integer q_j such that

$$t_j(f) \in F_{q_j}\mathbb{C}((\xi_j)) \setminus F_{q_j-1}\mathbb{C}((\xi_j)).$$

There exists at least one k with $q_k = q$. Moreover, we may assume

$$|\Phi\rangle \in \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N), \quad d_1 + \cdots + d_N = r.$$

Then, we have

$$\rho_j(X \otimes t_j(f))|\Phi\rangle \in \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_j}(d_j + q_j) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N).$$

Hence, we conclude

$$(X \otimes f)|\Phi\rangle \in F_{r+q}\mathcal{H}_{\vec{\lambda}} \setminus F_{r+q-1}\mathcal{H}_{\vec{\lambda}}.$$

Since $r + q > p$, this is a contradiction. □

Now we are ready to prove Theorem 3.1.5. Fix a positive integer m satisfying

$$m \geqslant \frac{1}{N}(2g(C) - 1) + 1.$$

By Lemma 3.1.9, we have

$$Gr^F_\bullet(\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}) = Gr^F_\bullet(\widehat{\mathfrak{g}}(\mathfrak{X}))Gr^F_\bullet(\mathcal{H}_{\vec{\lambda}}).$$

Therefore, if we introduce the induced filtration $\{F_\bullet\}$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$, we have

$$Gr^F_\bullet\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = Gr^F_\bullet(\mathcal{H}_{\vec{\lambda}})/Gr^F_\bullet(\widehat{\mathfrak{g}}(\mathfrak{X}))Gr^F_\bullet(\mathcal{H}_{\vec{\lambda}}).$$

Since we have that

$$Gr^F_\bullet\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\vec{\lambda}}$$

by Corollary 3.1.8, there is a surjective homomorphism

$$V_m = \mathcal{H}_{\vec{\lambda}} \Big/ \mathfrak{g} \otimes_{\mathbb{C}} \Big(\bigoplus_{j=1}^{N} \mathbb{C}[\xi_j^{-1}]\xi_j^{-m}\Big) \twoheadrightarrow Gr^F_\bullet\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}).$$

Now, by Proposition 3.1.6, V_m is a finite-dimensional vector space. Hence, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is also a finite-dimensional vector space. This finishes the proof of Theorem 3.1.5. □

Recently Suzuki [34] has given an elementary proof of Theorem 3.1.5 without using Gabber's Theorem.

For the later purpose we need to define the space of conformal blocks for non-connected curves. For an N_1-pointed stable curve with formal neighbourhoods

$$\mathfrak{X}_1 = (C_1; P_1, \ldots, P_{N_1}; s_1, \ldots, s_{N_1})$$

and an N_2-pointed stable curve with formal neighbourhoods

$$\mathfrak{X}_2 = (C_2; Q_1, \ldots, Q_{N_2}; t_1, \ldots, t_{N_2})$$

the connected sum $\mathfrak{X} = \mathfrak{X}_1 \bigsqcup \mathfrak{X}_2$ is defined as

$$\mathfrak{X} = (C; P_1, \ldots, P_{N_1}, Q_1, \ldots, Q_{N_1}; s_1, \ldots, s_{N_1}, t_1, \ldots, t_{N_2}),$$

where $C = C_1 \sqcup C_2$. The data $\mathfrak{X}$ satisfies all the conditions of Definition 1.1.1 and Definition 1.1.3 except connectedness of the curve C. For such $\mathfrak{X}$ we can define the space of conformal blocks $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and the space of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$. It is easy to show the following Proposition.

Proposition 3.1.10. *For a non-connected N-pointed stable curve with formal neighbourhoods*

$$\mathfrak{X} = \mathfrak{X}_1 \bigsqcup \mathfrak{X}_2$$

there are canonical isomorphisms

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}_{\vec{\lambda}_1}(\mathfrak{X}_1) \otimes \mathcal{V}_{\vec{\lambda}_2}(\mathfrak{X}_2),$$
$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathfrak{X}_1) \otimes \mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathfrak{X}_2),$$

where $\vec{\lambda} = (\vec{\lambda}_1, \vec{\lambda}_2)$.

3.2. Formal Neighbourhoods.

Let us examine the dependence of the definition of the conformal blocks on the formal neighbourhoods. For that purpose we introduce some notation. We let $\mathcal{D}$ be the automorphism group $\mathrm{Aut}\,\mathbb{C}((\xi))$ of the field $\mathbb{C}((\xi))$. The group $\mathcal{D}$ may be regarded as the automorphism of the ring $\mathbb{C}[[\xi]]$. There is a natural isomorphism

$$\mathcal{D} \simeq \Big\{ \sum_{n=0}^{\infty} a_n \xi^{n+1} \Big| a_0 \neq 0 \Big\},$$
$$h \mapsto h(\xi),$$

where the composition $h_1 \circ h_2$ of h_1, $h_2 \in \mathcal{D}$ corresponds to the formal power series $h_1(h_2(\xi))$. By this isomorphism we identify $\mathcal{D}$ with the set of formal power series without constant terms.

We introduce a filtration

$$\mathcal{D} = \mathcal{D}^0 \supset \mathcal{D}^1 \supset \mathcal{D}^2 \supset \dots$$

by putting

$$\mathcal{D}^p = \{ h \in \mathcal{D} \mid h(\xi) = \xi + a_p \xi^{p+1} + \dots \}$$

for a positive integer $p \geqslant 1$. Also put

$$\underline{d} = \mathbb{C}[[\xi]]\xi \frac{d}{d\xi},$$
$$\underline{d}^p = \mathbb{C}[[\xi]]\xi^{p+1} \frac{d}{d\xi},$$

for a positive integer p. We have a decreasing filtration

$$\underline{d} = \underline{d}^0 \supset \underline{d}^1 \supset \underline{d}^2 \supset \dots .$$

For any $\underline{\ell} \in \underline{d}$ and $f(\xi) \in \mathbb{C}[[\xi]]$, define $\exp(\underline{\ell})(f(\xi))$ by

$$\exp(\underline{\ell})(f(\xi)) = \sum_{k=0}^{\infty} \frac{1}{k!} (\underline{\ell}^k f(\xi)).$$

This is well-defined and $\exp(\underline{\ell})$ is an automorphism of $\mathbb{C}[[\xi]]$ and $\mathbb{C}((\xi))$. For example, for $\underline{\ell} = \alpha\xi\frac{d}{d\xi}$, $\alpha \in \mathbb{C}^*$, we have

$$\exp(\underline{\ell})(\xi) = e^{\alpha}\xi,$$

hence $\exp(\underline{\ell})$ is an automorphism of $\mathbb{C}((\xi))$ given by multiplying ξ by the non-zero e^{α}. In particular, $\exp\left(2\pi\sqrt{-1}n\xi\frac{d}{d\xi}\right)$ is the identity automorphism. It is easy to show the following Lemma.

Lemma 3.2.1. *The exponential mapping*

$$\exp : \underline{d} \longrightarrow \mathcal{D}$$
$$\underline{\ell} \longmapsto \exp(\underline{\ell})$$

is surjective. Moreover, for each positive integer p,

$$\exp : \underline{d}^p \longrightarrow \mathcal{D}^p$$

induces an isomorphism.

The above example shows that the exponential mapping is not injective on the $\underline{d}^0$-level.

For any $\underline{\ell} \in \underline{d}^p$, $p \geqslant 1$, define $\exp(T[\underline{\ell}])$ by

$$\exp(T[\underline{\ell}]) = \sum_{k=0}^{\infty} \frac{1}{k!} T[\underline{\ell}]^k.$$

The operator $\exp(T[\underline{\ell}])$ acts on $\mathcal{H}_\lambda$ from the left and on $\mathcal{H}_\lambda^\dagger$ from the right, and induces the identity operator on $Gr_\bullet^F \mathcal{H}_\lambda$ and on $Gr_F^\bullet \mathcal{H}_\lambda^\dagger$.

The following results can be obtained by direct calculations.

Definition 3.2.3. For any automorphism $h \in \mathcal{D}^p$, $p \geqslant 1$, the operator $G[h]$ on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$ is defined by

$$G[h] = \exp(-T[\underline{\ell}]),$$

where

$$h = \exp(\underline{\ell}).$$

Note that $G[h]$ is an automorphism of $\mathcal{H}_\lambda$ and of $\mathcal{H}_\lambda^\dagger$. The following theorem plays an essential role in the our arguments.

Theorem 3.2.4. *For $h \in \mathcal{D}^p$, $p \geqslant 1$ and $f(\xi) \in \mathbb{C}((\xi))$ we have the following equalities as operators on $\mathcal{H}_\lambda$ and on $\mathcal{H}_\lambda^\dagger$.*

(1) $G[h](X \otimes f(\xi))G[h]^{-1} = X \otimes f(h(\xi))$,
(2) $G[h_1] \cdot G[h_2] = G[h_1 \circ h_2]$, *for* $h_1, h_2 \in \mathcal{D}^p$, $p \geqslant 1$,
(3) $G[h]T[\underline{\ell}]G[h]^{-1} = T[\mathrm{ad}(h)(\underline{\ell})] + \frac{c_v}{12}\mathrm{Res}_{\xi=0}(\{h(\xi);\xi\}\ell(\xi)d\xi)$,

where $\underline{\ell} = \ell(\xi)\frac{d}{d\xi} \in \mathbb{C}((\xi))\frac{d}{d\xi}$, $f \in \mathbb{C}((\xi))$ and $X \in \mathfrak{g}$.

From this theorem we infer the following theorem.

Theorem 3.2.5. *For any $h_j \in \mathcal{D}^p$, $p \geqslant 1$, $j = 1, 2, \ldots N$ and N-pointed curve $\mathfrak{X} = (C; Q_1, \ldots, Q_N, \xi_1, \ldots, \xi_N)$ with formal neighbourhoods put*

$$\mathfrak{X}_{(h)} = (C; Q_1, \ldots, Q_N, h_1(\xi_1), \ldots, h_N(\xi_N)).$$

Then, the isomorphism $G[h_1]^{-1}\widehat{\otimes}\cdots\widehat{\otimes}G[h_N]^{-1}$

$$\begin{aligned}\mathcal{H}_{\vec{\lambda}}^\dagger = \mathcal{H}_{\lambda_1}^\dagger\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\lambda_N}^\dagger &\longrightarrow \mathcal{H}_{\vec{\lambda}}^\dagger = \mathcal{H}_{\lambda_1}^\dagger\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\lambda_N}^\dagger,\\ \langle\phi_1\widehat{\otimes}\phi_2\widehat{\otimes}\cdots\widehat{\otimes}\phi_N| &\longmapsto \langle\phi_1 G[h_1]^{-1}\widehat{\otimes}\phi_2 G[h_2]^{-1}\widehat{\otimes}\cdots\widehat{\otimes}\phi_N G[h_N]^{-1}|\end{aligned}$$

induces an isomorphism

$$G[h_1]^{-1}\widehat{\otimes}\cdots\widehat{\otimes}G[h_N]^{-1}\colon \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}) \longrightarrow \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}_{(h)}).$$

Proof. Note that we have

$$\begin{aligned}\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}) = \Big\{ \langle\Psi| \in \mathcal{H}_{\vec{\lambda}}^\dagger \;\Big|\; &\sum_{j=1}^N \langle\Psi|\rho_j\left(X \otimes f(\xi_j)\right) = 0,\\ &\text{for any } f \in H^0\left(C, \mathcal{O}_{\mathcal{C}}\left(*\sum Q_j\right)\right)\Big\}\\ \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}_{(h)}) = \Big\{ \langle\Psi| \in \mathcal{H}_{\vec{\lambda}}^\dagger \;\Big|\; &\sum_{j=1}^N \langle\Psi|\rho_j\left(X \otimes f\left(h_j(\xi_j)\right)\right) = 0,\\ &\text{for any } f \in H^0\left(C, \mathcal{O}_{\mathcal{C}}\left(*\sum Q_j\right)\right)\Big\}\end{aligned}$$

and $X \otimes f(h(\xi_j)) = G[h](X \otimes f(\xi))G[h]^{-1}$. Then, for $\langle\Psi| \in \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X})$, we have

$$\langle\Psi|G[h_1]^{-1}\widehat{\otimes}\cdots\widehat{\otimes}G[h_N]^{-1} \in \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}_{(h)}),$$

since

$$\begin{aligned}&\sum_{j=1}^N (\langle\Psi|G[h_1]^{-1}\widehat{\otimes}\cdots\widehat{\otimes}G[h_N]^{-1})\rho_j(X \otimes f(h(\xi))\\ &\qquad = \Big(\sum_{j=1}^N \rho_j\left(X \otimes f\left(h(\xi)\right)\right)\Big)G[h_1]\widehat{\otimes}\cdots\widehat{\otimes}G[h_N] = 0.\end{aligned}$$

□

Corollary 3.2.6. *If two N-pointed curves with formal neighbourhoods*

$$\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$$

and

$$\mathfrak{X}' = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$$

induce the same N-pointed curve with first order neighborhoods, then there exists a canonical isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}').$$

Proof. By the assumption there is $h_j \in \mathcal{D}^1$ with $\eta_j = h_j(\xi_j)$. Hence, there is an isomorphism induced by $G[h_1]^{-1} \widehat{\otimes} \ldots \widehat{\otimes} G[h_N]^{-1}$. $\square$

3.3. Basic Properties of the Space of Conformal Blocks.

In this section we shall show basic properties of the space of conformal blocks. First we shall prove the so called *propagation of vacua* which plays an important role in describing the space of vacua.

For $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ let P be a non-singular point of the curve C and η a formal parameter of C at P. Put

$$\widetilde{\mathfrak{X}} = (C; Q_1, \ldots, Q_N, Q_{N+1}; \eta_1, \ldots, \eta_N, \eta_{N+1})$$

where $Q_{N+1} = P$ and $\eta_{N+1} = \eta$.

In the following we fix a highest weight vector $|0\rangle$ of the integrable left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_0$. Since there is a canonical inclusion

$$\begin{aligned} \mathcal{H}_{\vec{\lambda}} &\longrightarrow \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0 \\ |v\rangle &\longmapsto |v\rangle \otimes |0\rangle, \end{aligned}$$

we have a canonical surjection

$$\widehat{\iota}^* \colon \mathcal{H}^\dagger_{\vec{\lambda}} \widehat{\otimes} \mathcal{H}^\dagger_0 \longrightarrow \mathcal{H}^\dagger_{\vec{\lambda}}.$$

Theorem 3.3.1. *The canonical surjection $\widehat{\iota}^*$ induces a canonical isomorphism*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Proof. For an element $\langle \widetilde{\Psi}| \in \mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$ put $\langle \Psi| = \widehat{\iota}^*(\langle \widetilde{\Psi}|) \in \mathcal{H}^\dagger_{\vec{\lambda}}$. Choose $X \in \mathfrak{g}$, $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$ and $f \in H^0\big(C, \mathcal{O}_C\big(* \sum_{j=1}^N Q_j\big)\big)$. Then by our definition we have

$$\sum_{j=1}^N \langle \Psi | \rho_j(X[f]) | u \rangle = \sum_{j=1}^N \langle \widetilde{\Psi} | \rho_j(X[f]) | u \otimes 0 \rangle.$$

On the other hand, since f is regular at the point $Q_{N+1} = P$, we have

$$\langle \widetilde{\Psi} | \rho_{N+1}(X[f]) | u \otimes 0 \rangle = 0.$$

Hence we have

$$\sum_{j=1}^N \langle \widetilde{\Psi} | \rho_j(X[f]) | u \otimes 0 \rangle = \sum_{j=1}^{N+1} \langle \widetilde{\Psi} | \rho_j(X[f]) | u \otimes 0 \rangle = 0.$$

Thus we have $\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ and we have a linear mapping

$$\iota^*\colon \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \longrightarrow \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}).$$

First we shall show that the linear mapping ι^* is injective.

Assume that $\langle\Psi| = \iota^*(\langle\widetilde{\Psi}|) = 0$. By induction on p we show that

$$\langle\widetilde{\Psi}|u \otimes v\rangle = 0, \quad \text{for all } u \in \mathcal{H}_{\vec{\lambda}} \text{and } v \in F_p\mathcal{H}_0. \tag{3.3.1}$$

By our assumption we have

$$\langle\widetilde{\Psi}|u \otimes 0\rangle = \langle\Psi|u\rangle = 0.$$

Hence (3.3.1) is true for $p = 0$. Next assume that (3.3.1) holds for p. Choose an element $X(m)|v\rangle \in F_{p+1}\mathcal{H}_0$, where $|v\rangle \in F_p\mathcal{H}_0$. Choose a meromorphic function $f \in H^0\big(C, \mathcal{O}_C(*\sum_{j=1}^{N+1} Q_j)\big)$ and a positive integer M such that at the point P we have

$$f \equiv \eta^m \mod (\eta^M) \tag{3.3.2}$$

and such that

$$X \otimes \eta^k|v\rangle = 0 \quad \text{for all } \ k \geqslant M. \tag{3.3.3}$$

Then we have

$$\begin{aligned}\langle\widetilde{\Psi}|u \otimes X(m)|v\rangle &= \langle\widetilde{\Psi}|u \otimes (X[f])v\rangle \\ &= -\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[f])u \otimes v\rangle \\ &= 0,\end{aligned}$$

since by the induction hypothesis $\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[f])u \otimes v\rangle = 0$. Thus (3.3.1) holds for $p+1$. Hence $\langle\widetilde{\Psi}|u \otimes v\rangle = 0$ for any $|u \otimes v\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$. Hence, $\langle\widetilde{\Psi}| = 0$.

Next we shall show that ι^* is surjective. For that purpose, to a given $\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ we attach an element $\langle\widetilde{\Psi}| \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{M}_0, \mathbb{C})$, where $\mathcal{M}_0$ is the Verma module of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with the trivial representation of $\mathfrak{g}$. (See (2.1.4).)

The linear functional $\langle\widetilde{\Psi}|$ is defined inductively as a linear mapping of $\mathcal{H}_{\vec{\lambda}} \otimes F_p\mathcal{M}_0$ to $\mathbb{C}$ as follows. First define

$$\langle\widetilde{\Psi}|u \otimes 0\rangle = \langle\Psi|u\rangle \quad \text{for any } u \in \mathcal{H}_{\vec{\lambda}}.$$

Then we have

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[g])|u \otimes 0\rangle = \sum_{j=1}^{N}\langle\Psi|\rho_j(X[g])|u\rangle = 0,$$

for any element $g \in H^0\big(C, \mathcal{O}_C\big(*\sum_{j=1}^{N} Q_j\big)\big)$.

Now assume that $\langle\widetilde{\Psi}|$ is defined as a linear mapping of $\mathcal{H}_{\vec{\lambda}} \otimes F_p\mathcal{M}_0$ to $\mathbb{C}$ with

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[g])|u \otimes v\rangle = 0, \tag{3.3.4}$$

for any $|u \otimes v\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes F_p\mathcal{M}_0$ and $g \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. Then, on $\mathcal{H}_{\vec{\lambda}} \otimes F_{p+1}\mathcal{M}_0$ the linear mapping $\langle\widetilde{\Psi}|$ is defined by

$$\langle\widetilde{\Psi}|u \otimes X(m)v\rangle = -\sum_{j=1}^{N}\langle\widetilde{\Psi}|(\rho_j(X[f])u \otimes v\rangle, \quad \text{for any } u \in \mathcal{H}_{\vec{\lambda}}, \tag{3.3.5}$$

where $X(m)|v\rangle \in F_{p+1}\mathcal{M}_0$, $|v\rangle \in F_p\mathcal{M}_0$ and a meromorphic function f is chosen in the same way as in (3.3.2) and (3.3.3). It is easy to show that this is well-defined and has the property

$$\sum_{j=1}^{N+1} \langle\widetilde{\Psi}|\rho_j(X[g]) = 0 \quad \text{on} \quad \mathcal{H}_{\vec{\lambda}} \otimes F_{p+1}\mathcal{M}_0,$$

for each element $g \in H^0\big(C, \mathcal{O}_C\big(*\sum_{j=1}^{N+1} Q_j\big)\big)$. A straightforward calculation shows the equality

$$\begin{aligned}\langle\widetilde{\Psi}|u\otimes X(m_1)Y(m_2)v\rangle &- \langle\widetilde{\Psi}|u\otimes Y(m_2)X(m_1)v\rangle \\ &= \langle\widetilde{\Psi}|u\otimes([X,Y](m_1+m_2)+\ell\cdot(X,Y)m_1\delta_{m_1+m_2,0})v\rangle,\end{aligned}$$

as long as $X(m_1)Y(m_2)|v\rangle \in F_{p+1}\mathcal{M}_0$ and $|v\rangle \in F_{p+1}\mathcal{M}_0$. This equality shows that $\langle\widetilde{\Psi}|$ is defined as a linear mapping on $\mathcal{H}_{\vec{\lambda}}$. To show that $\langle\widetilde{\Psi}|$ descends to a linear form on $\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_0$, it is enough to show the equality

$$\langle\widetilde{\Psi}|u\otimes X_\theta(-1)^{\ell+1}|0\rangle = 0. \tag{3.3.6}$$

Let $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$ be a meromorphic function on C which has a zero of order 1 at $P = Q_{N+1}$ and which satisfies (3.3.2) with $m = 1$ and $M \geq 2$. By Proposition 2.2.11 there is a positive integer n depending on $|u\rangle$ such that for any j, $j = 1, \ldots, N$, we have

$$\rho_j(X_{-\theta}[f]^k)|u\rangle = 0, \quad \text{if } k \geq n/N. \tag{3.3.7}$$

At the point $P = Q_{N+1}$ put

$$E = X_{-\theta}[f], \quad F = X_\theta(-1), \quad H = [E,F].$$

Then, we have

$$H = [X_{-\theta}, X_\theta][\eta^{-1}f] + (X_{-\theta}, X_\theta)c = -H_\theta[\eta^{-1}f] + c.$$

Since $\eta^{-1}f$ is holomorphic at the point P, we have

$$H|0\rangle = \ell|0\rangle.$$

Moreover, by a simple calculation we have

$$HF^k|0\rangle = (\ell - 2k)F^k|0\rangle, \quad EF^k|0\rangle = (k\ell - k^2 + k)F^{k-1}|0\rangle.$$

Hence, for a postive integer m we have

$$E^m F^{m+\ell+1}|0\rangle = c_m F^{\ell+1}|0\rangle,$$

where c_m is a non-zero constant. Now by (3.3.5) and (3.3.7), we obtain

$$\begin{aligned} c_n\langle\widetilde{\Psi}|u\otimes X_\theta(-1)^{\ell+1}|0\rangle &= \langle\widetilde{\Psi}|u\otimes(X_{-\theta}[f])^n X_\theta(-1)^{n+\ell+1}|0\rangle \\ &= (-1)^n \sum_{n_1+\ldots+n_N=n} \frac{n!}{n_1!n_2!\ldots n_N!}\times \\ &\qquad \Big\langle\widetilde{\Psi}\Big|\prod_{j=1}^{N}\rho_j(X_{-\theta}[f])^{n_j}u\otimes X_\theta(-1)^{\ell+1}\Big|0\Big\rangle \\ &= 0. \end{aligned}$$

Thus we obtain the desired result. □

Corollary 3.3.2. *There is a canonical isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$$

Remark 3.3.3. Theorem 3.3.1 and Corollary 3.3.2 say that in the study of the space of vacua and covacua attached to an N-pointed stable curve with formal neighbourhoods we can add as many points with formal neighbourhoods as we need. Therefore, as we mentioned above, we can always assume that the condition (Q) is satisfied. Below this fact will be often used and play an essential role in proving important theorems.

The proof of the above theorem relies on the following theorem.

Theorem 3.3.4. *The canonical inclusion*

$$V_{\lambda_1} \otimes \ldots V_{\lambda_{N-1}} \otimes \mathcal{H}_{\lambda_N} \hookrightarrow \mathcal{H}_{\vec{\lambda}}$$

induces a natural projection

$$p\colon \mathcal{V}^{\dagger}_{\vec{\lambda}} \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \ldots V_{\lambda_{N-1}} \otimes \mathcal{H}_{\lambda_N}, \mathbb{C}).$$

The image is contained in

$$V^{\dagger}_{\vec{\lambda},0} = \{\, \langle \Phi| \in V_{\lambda_1} \otimes \ldots V_{\lambda_{N-1}} \otimes \mathcal{H}_{\lambda_N} \mid \langle \Phi| X \otimes f = 0, \text{ for any } f \in H^0(C, \mathcal{O}_C(*Q_N)) \,\}$$

and the map $p\colon \mathcal{V}^{\dagger}_{\vec{\lambda}} \to V^{\dagger}_{\vec{\lambda},0}$ *is injective.*

For an element $\mu \in P_\ell$ put

$$\mu^{\dagger} = -w(\mu),$$

where w is the longest element of the Weyl group of the simple Lie algebra $\mathfrak{g}$ (in other word, $w(\Delta_+) = \Delta_-$). Note that $\mu^{\dagger}$ is also characterized by the fact that $-\mu^{\dagger}$ is the lowest weight of the $\mathfrak{g}$-module V_μ. For example, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, we have $\mu^{\dagger} = \mu$.

For an N-pointed stable curve $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ with formal neighbourhoods, assume that the curve C has a double point P. Let $\nu\colon \widetilde{C} \to C$ be the normalization at the point P. Put $\nu^{-1}(P) = \{P', P''\}$. Furthermore we introduce formal neighbourhoods η' and η'' at P' and P'', respectively.

In the proof of the following Theorem 3.3.5 we shall use the results of Theorem 3.4.1. We shall not use Theorem 3.3.5 in the proof of Theorem 3.4.1.

Theorem 3.3.5. *Using the above notation, there is a canonical isomorphism*

$$\bigoplus_{\mu \in P_l} \mathcal{V}^{\dagger}_{\mu, \mu^{\dagger}, \vec{\lambda}}(\widetilde{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}).$$

Proof. The diagonal action of $\mathfrak{g}$ on $V_\mu \otimes V_{\mu^{\dagger}}$ makes $V_\mu \otimes V_{\mu^{\dagger}}$ a $\mathfrak{g}$-module and it contains the trivial $\mathfrak{g}$-module with multiplicity one. Let $|0_{\mu,\mu^{\dagger}}\rangle$ be a basis of the trivial $\mathfrak{g}$-submodule of $V_\mu \otimes V_{\mu^{\dagger}}$. That is,

$$\rho_1(X)|0_{\mu,\mu^{\dagger}}\rangle + \rho_2(X)|0_{\mu,\mu^{\dagger}}\rangle = 0,$$

for all $X \in \mathfrak{g}$.

For example, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\mu = m\alpha$, we let e_0 be a highest weight vector, then $He_0 = \frac{m}{2} e_0$. Put

$$e_j = \frac{1}{j!} F^j e_0, \quad j = 1, 2, \ldots, m.$$

Then, we may choose

$$|0_{\mu,\mu^\dagger}\rangle := \sum_{j=0}^{m}(-1)^m \frac{j!(m-j)!}{m!} e_j \otimes e_{m-j}.$$

Now the module $\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$ contains a subspace

$$\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}} \supset |0_{\mu,\mu^\dagger}\rangle \otimes \mathcal{H}_{\vec{\lambda}} \simeq \mathcal{H}_{\vec{\lambda}}.$$

For any element $\langle\widetilde{\Psi}| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, define $\langle\Psi| \in \mathcal{H}^\dagger_{\vec{\lambda}}$ by

$$\langle\Psi|\Phi\rangle = \langle\widetilde{\Psi}|0_{\mu,\mu^\dagger} \otimes \Phi\rangle, ,$$

for all $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$. Then, for any meromorphic function

$$f \in H^0\Big(C, \mathcal{O}_C\Big(*\sum_{j=1}^{N} Q_j\Big)\Big)$$

we have

$$\begin{aligned}\sum_{j=1}^{N}\langle\Psi|\rho_j(X[f])|\Phi\rangle &= \sum_{j=1}^{N}\langle\widetilde{\Psi}|0_{\mu,\mu^\dagger} \otimes \rho_j(X[f])\Phi\rangle \\ &= \sum_{j=1}^{N+2}\langle\widetilde{\Psi}|\rho_j(X[f])|0_{\mu,\mu^\dagger} \otimes \Phi\rangle = 0,\end{aligned}$$

since if we regard f as a meromorphic function on $\widetilde{C}$, we have $f(P') = f(P'')$ and $\rho_{P'}(X[f])|0_{\mu,\mu^\dagger}\rangle + \rho_{P''}(X[f])|0_{\mu,\mu^\dagger}\rangle = 0$. Hence we have

$$\sum_{j=1}^{N}\langle\Psi|\rho_j(X[f]) = 0, \quad \text{for any } \ f \in H^0\Big(C, \mathcal{O}_C\Big(*\sum_{j=1}^{N} Q_j\Big)\Big).$$

Thus we have a canonical $\mathbb{C}$-linear mapping

$$\iota_\mu \colon \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \longrightarrow \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Let us show that the mapping ι_μ is injective. For that purpose, we first show that for $\langle\Psi| = \iota_\mu(\langle\widetilde{\Psi}|)$, $\langle\widetilde{\Psi}| \in \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, we have

$$\langle\Psi|X(P)|\Phi\rangle dP = \langle\widetilde{\Psi}|X(P)|0_{\mu,\mu^\dagger} \otimes \Phi\rangle dP. \tag{3.3.8}$$

Note that by Claim 3 of the proof of Theorem 3.4.1, the expansion of the left hand side of (3.3.8) at Q_j with respect to the formal parameter ξ_j has the form

$$\sum_{n\in\mathbb{Z}}\langle\Psi|\rho_j(X(n))|\Phi\rangle\xi_j^{-n-1}d\xi_j.$$

Similarly the right hand side of (3.3.8) has the expansion

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}|\rho_j(X(n))|0_{\mu,\mu^\dagger} \otimes \Phi\rangle\xi_j^{-n-1}d\xi_j \\ &\qquad = \sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}|(0_{\mu,\mu^\dagger}) \otimes \rho_j(X(n))\Phi\rangle\xi_j^{-n-1}d\xi_j \\ &\qquad = \sum_{n\in\mathbb{Z}}\langle\Psi|\rho_j(X(n))|\Phi\rangle\xi_j^{-n-1}d\xi_j.\end{aligned}$$

Hence the equality (3.3.8) holds. Similar argument shows the equality

$$\langle\Psi|X_1(P_1)\dots X_M(P_M)|\Phi\rangle dP_1\dots dP_M = \langle\widetilde{\Psi}|X_1(P_1)\dots X_M(P_M)|0_{\mu,\mu^\dagger}\otimes\Phi\rangle dP_1\dots dP_M.$$

Now assume that $\langle\Psi| = 0$. By Theorem 3.4.1, (3) we have

$$\langle\widetilde{\Psi}|X_2(P_2)\dots X_M(P_M)|\rho_{P'}(X_1(n))0_{\mu,\mu^\dagger}\otimes\Phi\rangle = 0.$$

Applying again Theorem 3.4.1, (3), we obtain

$$\begin{aligned}\langle\widetilde{\Psi}|\rho_{P'}(X_2(n_2)X_1(n_1))0_{\mu,\mu^\dagger}\otimes\Phi\rangle &= 0,\\ \langle\widetilde{\Psi}|\rho_{P'}(X_1(n_1))\rho_{P''}(X_2(n_2))0_{\mu,\mu^\dagger}\otimes\Phi\rangle &= 0,\\ \langle\widetilde{\Psi}|\rho_{P''}(X_1(n_1)X_2(n_2))0_{\mu,\mu^\dagger}\otimes\Phi\rangle &= 0.\end{aligned}$$

Repeating the same process we can show that

$$\langle\widetilde{\Psi}|\widetilde{\Phi}\rangle = 0, \quad \text{for any } \widetilde{\Phi}\in\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}},$$

since $\mathcal{H}_\mu\otimes\mathcal{H}_{\mu^\dagger}$ is an irreducible $\widehat{\mathfrak{g}}\times\widehat{\mathfrak{g}}$-module. Hence ι_μ is injective. Let us consider the $\mathbb{C}$-linear homomorphism

$$\iota\colon \bigoplus_{\mu\in P_\ell}\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \xrightarrow{\oplus\iota_\mu} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

We shall show that ι is injective. For this purpose, to the points P' and P'' we associate certain right $\mathfrak{g}$-modules and integrable right $\widehat{\mathfrak{g}}$-modules.

Fix an element $\langle\Psi|\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. Let h be a meromorphic function on $\widetilde{C}$ such that

$$\begin{aligned}h &\in H^0\Big(\widetilde{C},\mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^N Q_j)\Big),\\ h(P') &= 1,\\ h(P'') &= 0.\end{aligned}\tag{3.3.9}$$

If h' also satisfies the properties (3.3.9), then $h-h'$ can be regarded as a meromorphic function on C and $h-h'\in H^0\big(C,\mathcal{O}_C(*\sum_{j=1}^N Q_j)\big)$. Hence, by the gauge condition for each $|u\rangle\in\mathcal{V}_{\vec{\lambda}}$,

$$\sum_{j=1}^N\langle\Psi|\rho_j(X[h])|u\rangle$$

is independent of the choice of a meromorphic function h satisfying (3.3.9). For each element $X\in\mathfrak{g}$ define $\langle\Psi|\rho_{P'}(X)\in\mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}},\mathbb{C})$ by

$$\langle\Psi|\rho_{P'}(X)|u\rangle = -\sum_{j=1}^N\langle\Psi|\rho_j(X[h])|u\rangle, \qquad |u\rangle\in V_{\vec{\lambda}},$$

where h satisfies (3.3.9). This is well-defined.

Next for $X,Y\in\mathfrak{g}$ define $\langle\Psi|\rho_{P'}(X)\rho_{P'}(Y)\in\mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}},\mathbb{C})$ by

$$\langle\Psi|\rho_{P'}(X)\rho_{P'}(Y)|u\rangle = \sum_{j_1=1,j_2=1}^N\langle\Psi|\rho_{j_2}(Y[h_2])\rho_{j_1}(X[h_1])|u\rangle, \quad \text{for all } |u\rangle\in V_{\vec{\lambda}},$$

where h_1 and h_2 satisfy (3.3.9). The definition is independent of the choice of h_2 by the same reason as above. That the definition is independent of the choice of h_1 is proved as follows. Since $h_2 dh_1$ is a meromorphic one form on $\widetilde{C}$ having poles only at $Q_1, \dots, Q_N$, we have $\sum_{j=1}^N \mathrm{res}_{Q_j}(h_2 dh_1) = 0$. Therefore, we have the equality

$$\sum_{j_1,j_2=1}^N \langle \Psi | \rho_{j_2}(Y[h_2]) \rho_{j_1}(X[h_1]) | u \rangle$$
$$= \sum_{j_1,j_2=1}^N \langle \Psi | \rho_{j_1}(X[h_1]) \rho_{j_2}(Y[h_2]) | u \rangle - \sum_{j=1}^N \langle \Psi | \rho_j([X,Y][h_1 h_2]) | u \rangle.$$

The equality shows the independence of the choice of h_1, since $h_1 h_2$ also satisfies the properties (3.3.9). Moreover the above equality shows the equality

$$\langle \Psi | (\rho_{P'}(X)\rho_{P'}(Y) - \rho_{P'}(Y)\rho_{P'}(X)) = \langle \Psi | \rho_{P'}([X,Y]).$$

In this way we can define a right $\mathfrak{g}$-module $U(\langle \Psi |) \subset \mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C})$ at the point P'. In the same way we can construct a right $\mathfrak{g}$-module at the point P''.

We can also define an integrable right $\widehat{\mathfrak{g}}$-module $\widehat{U}(\langle \Psi |) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$. For example, $\langle \Psi | \rho_{P'}(X(n))$ is defined as follows. Let g be a meromorphic function on $\widetilde{C}$ such that

$$\begin{aligned} g &\in H^0\Big(\widetilde{C}, \mathcal{O}_{\widetilde{C}}\Big(*\Big(\sum_{j=1}^N Q_j + P'\Big)\Big)\Big), \\ g &\equiv \xi'^n \mod (\xi'^M) \quad \text{at} \quad P', \\ g(P'') &= 0, \end{aligned} \tag{3.3.10}$$

where $\xi' = \eta'^{-1}(\xi)$ is a formal parameter at the point P'. Define $\langle \Psi | \rho_{P'}(X(n))$ by

$$\langle \Psi | \rho_{P'}(X(n)) | u \rangle = -\sum_{j=1}^N \langle \Psi | \rho_j(X[g]) | u \rangle.$$

The definition is independent of the choice of a meromorphic function g satisfying (3.3.10). Similarly we can define $\langle \Psi | \rho_{P'}(X(n))\rho_{P'}(Y(m)) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$ and we have the equality

$$\begin{aligned} &\langle \Psi | (\rho_{P'}(X(n))\rho_{P'}(Y(m)) - \langle \Psi | \rho_{P'}(Y(m))\rho_{P'}(X(n))) \\ &\quad = \langle \Psi | \rho_{P'}([X,Y](m+n)) + \ell \cdot (X,Y) n \delta_{n+m,0} \langle \Psi |. \end{aligned} \tag{3.3.11}$$

In this way we can construct a right $\widehat{\mathfrak{g}}$-module $\widehat{U}(\langle \Psi |) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$. Since the action of $\rho_j(X_\alpha[g])$, $X_\alpha \in \mathfrak{g}_\alpha$, is locally nilpotent by Theorem 2.1.5, the action of $\rho_{P'}(X_\alpha(m))$ on $\widehat{U}(\langle \Psi |)$ is locally nilpotent. Hence $\widehat{U}(\langle \Psi |)$ is an integrable right $\widehat{\mathfrak{g}}$-module of level ℓ.

Since $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}))$ is finite dimensional, we can associate a finite right $\mathfrak{g}$-module to the point P'

$$U(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})) = \bigcup_{\langle \Psi | \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})} U(\langle \Psi |) \subset \mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C})$$

and an integrable right $\widehat{\mathfrak{g}}$-*module*

$$\widehat{U}(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})) = \bigcup_{\langle \Psi | \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})} \widehat{U}(\langle \Psi |) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$$

of level ℓ. By (3.3.11) we have an irreducible decomposition

$$U(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})) = \bigoplus_{\mu\in P_\ell} {V_{\mu^\dagger}}^{\oplus n_\mu}.$$

Then, by the remark given just above Lemma 2.2.10, we have

$$\widehat{U}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})) = \bigoplus_{\mu\in P_\ell} {\mathcal{H}^{\dagger}_{\mu}}^{\oplus n_\mu}. \tag{3.3.12}$$

We are now ready to prove the injectivity of ι. For an element $\langle\widetilde{\Psi}| \in \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, put $\langle\Psi| = \iota_\mu(\langle\widetilde{\Psi}|)$ and choose a meromorphic function h on $\widetilde{C}$ satisfying (3.3.9). Then we have

$$\begin{aligned}
&\langle\Psi|\rho_{P'}(X_1)\cdots\rho_{P'}(X_k)|u\rangle \\
&\qquad = (-1)^k \sum_{j_1=1,\dots,j_k=1}^{N} \langle\Psi|\rho_{j_1}(X_1[h])\cdots\rho_{j_k}(X_k[h])|u\rangle \\
&\qquad = (-1)^{k+1}\langle\widetilde{\Psi}|\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_k(0))|0_{\mu,\mu^\dagger}\otimes u\rangle.
\end{aligned}$$

Since the $\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_k(0))|0_{\mu,\mu^\dagger}\rangle$'s generate an irreducible left $\mathfrak{g}$-module isomorphic to $V_{\mu^\dagger}$, we conclude

$$U(\langle\Psi|) \subset V_{\mu}^{\dagger\oplus n_\mu}.$$

Hence, for $\langle\widetilde{\Psi}_\mu| \in \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$ and $\langle\widetilde{\Psi}_\nu| \in \mathcal{V}_{\nu,\nu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, $\mu \neq \nu$, we have

$$U(\langle\widetilde{\Psi}_\mu|) \cap U(\langle\widetilde{\Psi}_\nu|) = 0.$$

This means that ι is injective, since ι_μ is injective.

Finally let us prove that ι is surjective. By (3.3.12) for an element $\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ we have a decomposition

$$\langle\Psi| = \sum_{\mu\in P_\ell} \langle\Psi_\mu|, \qquad \langle\Psi_\mu| \in V_{\mu}^{\dagger\oplus n_\mu}.$$

We construct $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}},\mathbb{C})$ as follows. First note that $V_\mu \otimes V_{\mu^\dagger}$ is generated by elements

$$\rho_{P'}(X_1)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\cdots\rho_{P''}(Y_m)|0_{\mu,\mu^\dagger}\rangle,$$

$X_1,\dots,X_n,Y_1,\dots,Y_m \in \mathfrak{g}$. Moreover, $\langle\widetilde{\Psi}_\mu|$ defines a right $\widehat{\mathfrak{g}}$-module $\widehat{U}(\langle\widetilde{\Psi}_\mu|) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}},\mathbb{C})$. For each element $|v\rangle \in \mathcal{H}_{\vec{\lambda}}$ define

$$\langle\widetilde{\Psi}_\mu|0_{\mu,\mu^\dagger}\otimes v\rangle = \langle\Psi|v\rangle.$$

Define

$$\begin{aligned}
&\langle\widetilde{\Psi}_\mu|\rho_{P'}(X_1)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\cdots\rho_{P''}(Y_m)0_{\mu,\mu^\dagger}\otimes v\rangle \\
&\qquad = (-1)^m\langle\Psi_\mu|\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_n(0))\rho_{P'}(Y_m(0))\cdots\rho_{P'}(Y_1(0))v\rangle.
\end{aligned}$$

This is well-defined, since the diagonal action of $\mathfrak{g}$ on $\mathbb{C}|0_{\mu,\mu^\dagger}\rangle$ is trivial. This defines $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(V_\mu\otimes V_{\mu^\dagger}\otimes\mathcal{H}_{\vec{\lambda}},\mathbb{C})$. Now assume that we have already defined $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(F_p\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger}\otimes\mathcal{H}_{\vec{\lambda}},\mathbb{C})$ for non-negative integers p and q. Choose an element $\rho_{P'}(X(m))|u\otimes u'\otimes v\rangle \in$

$F_{p+1}\mathcal{H}_\mu \otimes F_q \otimes \mathcal{H}_{\mu^\dagger}$ with $|u \otimes u'\rangle \in F_p\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger}$. Choose a meromorphic function f on $\widetilde{C}$ such that

$$f \in H^0\Big(\widetilde{C}, \mathcal{O}_{\widetilde{C}}\Big(*\sum_{j=1}^{N} Q_j + *P' + *P''\Big)\Big),$$

$$f \equiv \xi'^m \mod (\xi'^M) \quad \text{at } P',$$

$$f \equiv 0 \mod (\xi''^M) \quad \text{at } P''.$$

Here we choose the positive integer M in such a way that

$$\rho_{P'}(X(n))|u\rangle = 0, \quad \rho_{P''}(X(n))|u\rangle = 0,$$

for all $n \geqslant M$. Then we define

$$\langle\widetilde{\Psi}_\mu|\rho_{P'}(X(m))|u \otimes u' \otimes v\rangle = -\sum_{j=1}^{N}\langle\widetilde{\Psi}_\mu|\rho_j(X[f])|u \otimes u' \otimes v\rangle.$$

By an argument similar to the proof of Proposition 3.3.4 we can show that the definition is independent of the choice of the meromorphic function f satisfying the above conditions and we have

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

Similarly we can define

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(F_p\mathcal{H}_\mu \otimes F_{q+1}\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

In this way we can show the existence of

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

Moreover, we can verify the gauge condition so that

$$\langle\widetilde{\Psi}_\mu| \in \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}).$$

By our construction we have $\iota_\mu(\langle\widetilde{\Psi}_\mu|) = \langle\Psi_\mu|$. $\square$

Corollary 3.3.5. *There is a canonical isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \xrightarrow{\sim} \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}).$$

Remark 3.3.6. In Chapter 4 we shall show that $\dim \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ depends only on the genus g of the curve and $\vec{\lambda}$. Therefore, by degenerating a smooth curve into a curve with nodes and using the above Theorem 3.3.5, the calculation of $\dim \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ is reduced the case of a curve of genus 0.

In case of an N-pointed smooth curve of genus 0, Theorem 3.3.4 can be replaced by the following stronger statement.

Proposition 3.3.7. *If the underlying curve of $\mathfrak{X}$ is $\mathbb{P}^1$, then the canonical mapping*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$$

is injective.

A proof will be given in section 3.5.

Note that Proposition 3.3.7 is not true, if the genus of the underlying curve is positive. For example, for a one-pointed elliptic curve

$$\mathfrak{X} = (E; Q, \zeta)$$

with a formal neighbourhood, we have

$$\dim \mathcal{V}^\dagger_0(\mathfrak{X}) = \ell + 1.$$

But $\mathrm{Hom}(V_0, \mathbb{C})$ is one-dimensional.

As a weak generalization of Proposition 3.3.7 we have the following Proposition.

Proposition 3.3.8. *If we have $N \geqslant g+1$, then the natural mapping*

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}\Big(F_1\Big(\bigotimes_{j=1}^{N}\mathcal{H}_{\lambda_j}\Big),\mathbb{C}\Big)$$

is injective.

3.4. Correlation Functions.

Let C be a semi-stable curve and ω_C its dualizing sheaf. Put

$$C^M = \overbrace{C\times\cdots\times C}^{M}.$$

Then C^M has singularities of codimension 1, but still we can define the dualizing sheaf ω_{C^M}, since C^M is locally a complete intersection. Moreover, we can show that

$$\omega_{C^M} = \omega_C^{\boxtimes M},$$

where $\pi_j\colon C^M \to C$ is the j-th projection and we define

$$\omega_C^{\boxtimes M} = \pi_1^*\omega_C \otimes \pi_2^*\omega_C \otimes\cdots\otimes \pi_M^*\omega_C.$$

Since C^M has singularities for a singular semi-stable curve, the (i,j)-th diagonal $\Delta_{ij} = \{(P_1,\dots,P_N)|P_i = P_j\}$ of C^M is only a Weil divisor and not a Cartier divisor. Let $\mathfrak{X} = (C;Q_1,\dots,Q_N;\eta_1,\dots,\eta_N)$ be an N-pointed stable curve with formal neighbourhoods.

Theorem 3.4.1. *Fix $\langle\Psi| \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$. For each non-negative integer M the data*

$$X_1, X_2, \dots, X_M \in \mathfrak{g}, \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$$

defines an element

$$F = \langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dP_1 dP_2 \dots dP_M$$

of

$$H^0\Big(C^M, \omega_C^{\boxtimes M}\Big(\sum_{1\leqslant i<j\leqslant M} *\Delta_{ij} + \sum_{i=1}^{M}\sum_{j=1}^{N} *\pi_i^{-1}(Q_j)\Big)\Big),$$

where $\Delta_{ij} = \{(P_1,\dots,P_N)|P_i = P_j\}$ is the diagonal. The meromorphic form has the following properties.

(0) *For $M=0$, $F = \langle\Psi|\Phi\rangle$ is the canonical pairing induced by* (3.1.12).

(1) *F is linear with respect to $|\Phi\rangle$ and multi-linear with respect to the X_i's.*

(2) *For any permutation $\sigma \in \mathfrak{S}_M$, we have*

$$F = \langle\Psi|X_{\sigma(1)}(P_{\sigma(1)})X_{\sigma(2)}(P_{\sigma(2)})\cdots X_{\sigma(M)}(P_{\sigma(M)})|\Phi\rangle dP_1 dP_2\dots dP_M.$$

For example, for a transposition $(i,i+1)$ we have

$$F = \langle\Psi|X_1(P_1)\cdots X_{i-1}(P_{i-1})X_{i+1}(P_{i+1})X_i(P_i) \\ X_{i+2}(P_{i+2})\dots X_M(P_M)|\Phi\rangle dP_1 dP_2\dots dP_M.$$

(3) *For $k=1,\dots,N$ and $\xi_k = \eta_k^{-1}(\xi)$ a holomorphic coordinate, we have the equality*

$$\operatorname*{Res}_{\xi_k=0}(\xi_k^n\langle\Psi|X(\xi_k)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle d\xi_k) \\ = \langle\Psi|X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\rho_k(X(n))\Phi\rangle.$$

In other words, we have an expansion

$$\langle\Psi|X(\xi_k)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle d\xi_k = \sum_{n\in\mathbb{Z}}\langle\Psi|X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\rho_k(X(n))\Phi\rangle\xi_k^{-n-1}d\xi_k$$

(4) *For a local holomorphic coordinate z at a nonsingular point P of the curve C, we have the following equality.*

$$\begin{aligned}\langle\Psi|X(P)Y(P')X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\Phi\rangle &= \frac{\ell\cdot(X,Y)}{(z(P)-z(P'))^2}\langle\Psi|X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\Phi\rangle \\ &\quad+\frac{1}{z(P)-z(P')}\langle\Psi|[X,Y](P')X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\Phi\rangle \\ &\quad+\text{regular at } P=P'.\end{aligned}$$

(5) *For a local holomorphic coordinate z at Q_i and for $|v\rangle\in V_{\vec\lambda}=V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_N}\subset\mathcal{H}_{\vec\lambda}$, we have an equality*

$$\begin{aligned}\langle\Psi|X(P)X_1(P_1)X_2(P_2)\cdots X_M(P_M)|v\rangle &= \frac{1}{z(P)-z(Q_i)}\langle\Psi|X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\rho_i(X)v\rangle \\ &\quad+\text{regular at } P=Q_i.\end{aligned}$$

These functions F are called *correlation functions of currents.*

Proof. Choose $M+1$ non-singular points $P_1,P_2,\dots,P_M,P$ of the curve C and their formal neighbourhoods $\eta_{N+1},\eta_{N+2},\dots,\eta_{N+M+1}$. Put

$$\widehat{\mathfrak{X}}=(C;Q_1,\dots,Q_N,Q_{N+1},\dots,Q_{N+M+1};\eta_1,\dots,\eta_{N+M+1}),$$
$$\widetilde{\mathfrak{X}}=(C;Q_1,\dots,Q_N,Q_{N+1},\dots,Q_{N+M};\eta_1,\dots,\eta_{N+M}),$$

where $Q_{N+i}=P_i$, $i=1,\dots,M$ and $Q_{N+M+1}=P$. By Theorem 3.3.1, there are canonical isomorphisms

$$\iota_M\colon\mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{X})\simeq\mathcal{V}^\dagger_{\vec\lambda,\vec0_M}(\widetilde{\mathfrak{X}}),$$
$$\iota_{M+1}\colon\mathcal{V}^\dagger_{\vec\lambda,\vec0_M}(\widetilde{\mathfrak{X}})\simeq\mathcal{V}^\dagger_{\vec\lambda,\vec0_{M+1}}(\widehat{\mathfrak{X}}),$$

where $\vec0_k=(\overbrace{0,\dots,0}^{k})$. For $\langle\Psi|\in\mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{X})$ put

$$\langle\widetilde\Psi|=\iota_M(\langle\Psi|),\quad\langle\widehat\Psi|=\iota_{M+1}(\langle\widetilde\Psi|).$$

Claim 1. *For any $|\widetilde u\rangle\in\mathcal{H}_{\vec\lambda}\otimes\mathcal{H}_{\vec0_M}$ and $X\in\mathfrak{g}$, $\langle\widehat\Psi|\widetilde u\otimes X(-1)|0\rangle d\eta$ defines a cotangent vector of the curve C at the point P.*

Proof. Choose a meromorphic function $f\in H^0(C,\mathcal{O}_C(*(P+Q_1))$ on C such that

$$f=\eta^{-1}+\text{regular at } P,$$
$$f\equiv0\mod(\xi_j^{n_j})\quad\text{at } Q_j,\ j\neq1,$$

where $\eta = \eta_{M+1}^{-1}(\xi)$, $\xi_j = \eta_j^{-1}(\xi)$ and n_j is sufficiently large so that

$$\rho_j(X[f])|\widetilde{u}\rangle = 0$$

and f is holomorphic at Q_j, $j \neq 1$. Then we have

$$\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle = \langle\widehat{\Psi}|\widetilde{u}\otimes(X[f])|0\rangle = -\langle\widehat{\Psi}|\rho_1(X[f])\widetilde{u}\otimes 0\rangle.$$

Hence, if we change the formal neighbourhood η_{M+1} to $\widetilde{\eta}_{M+1}$, we have

$$\widetilde{\eta} = \widetilde{\eta}_{M+1}^{-1}(\xi) = a_1\eta + a_2\eta^2 + \ldots, \quad a_1 \neq 0,$$
$$\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle_{\widetilde{\eta}} = a_1^{-1}\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle_{\eta}.$$

This implies that $\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle$ depends only on the first order infinitesimal neighbourhood and $\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle_\eta d\eta \in T_P^*C$ is independent of the choice of the formal coordinate.

Claim 2. *Put*

$$\omega_j = \sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}|\rho_j(X(n))|\widetilde{u}\rangle\xi_j^{-n-1}d\xi_j, \quad j = 1, 2, \ldots, N+M,$$

where $\xi_j = \eta_j^{-1}(\xi)$. *Then there is a meromorphic one form*

$$\omega \in H^0\Big(C, \omega_C\Big(*\sum_{j=1}^{N+M} Q_j\Big)\Big)$$

on C *such that*

$$t(\omega) = (\omega_1, \omega_2, \ldots, \omega_{N+M}),$$

where the mapping t *is defined in* (1.1.7).

Proof. For an element $f \in H^0\big(*\sum_{j=1}^{N+M} Q_j\big)$ let $f_j(\xi_j) = \sum a_n^{(j)}\xi_j^n$ be the formal Laurent expansion of f at the point Q_j by the formal parameter $\xi_j = \eta_j^{-1}(\xi)$. Hence $t(f) = (f_1(\xi_1), \ldots, f_{N+M}(\xi_{N+M}))$. Then we have

$$\begin{aligned}\sum_{j=1}^{N+M}\operatorname*{Res}_{\xi_j=0}(f_j(\xi_j)\omega_j) &= \sum_{j=1}^{N+M}\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}|\rho_j(X(n))|\widetilde{u}\rangle a_n^{(j)}\\ &= \langle\widetilde{\Psi}|X\otimes t(f)|\widetilde{u}\rangle = 0,\end{aligned}$$

since $\langle\Psi|X\otimes t(f) = 0$ by our assumption. Therefore, by Lemma 1.1.6 there exists an element $\omega \in H^0\big(C, \omega_C\big(*\sum_{j=1}^{N+M} Q_j\big)\big)$ with $t(\omega) = (\omega_1, \ldots, \omega_{N+M})$. This proves Claim 2.

Claim 3. *As cotangent vectors at* P *with formal parameter* η, *the vectors* $\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle d\eta$ *and* ω *are defined in Claim* 2 *coincide.*

In the following we express ω by

$$\omega = \langle\widetilde{\Psi}|X(P)|\widetilde{u}\rangle dP.$$

Proof. Since $\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle d\eta$ is a cotangent vector at P, we may assume that η is a local holomorphic coordinate of C at P. Choose a meromorphic function $f \in H^0(C, \mathcal{O}_C(*(P+Q_i))$ on C such that

$$f = \eta^{-1} + \text{regular at } P,$$
$$f \equiv 0 \mod(\xi_j^{n_j}) \quad \text{at } Q_j,\ j \neq i, 1 \leqslant j \leqslant N+M,$$

where n_j is sufficiently large so that $\rho_j(X[f])|\widetilde{u}\rangle = 0$ and $f\omega$ is holomorphic at Q_j, $j \neq i$, $1 \leqslant j \leqslant N+M$. Then we have

$$\begin{aligned}\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle &= -\sum_{k=1}^{N+M}\langle\widetilde{\Psi}|\rho_k(X[f])|\widetilde{u}\rangle \\ &= -\langle\widetilde{\Psi}|\rho_i(X[f])|\widetilde{u}\rangle.\end{aligned}$$

On the other hand, at the point P we have

$$\begin{aligned}\operatorname{res}_P\left(\frac{1}{\eta}\omega\right) &= \operatorname{res}_P(f\omega) \\ &= -\sum_{k=1}^{N+M}\operatorname{res}_{Q_k}(f\omega) \\ &= -\operatorname{res}_{Q_i}(f\omega) \\ &= -\operatorname*{Res}_{\xi_i=0}\Big(f_i(\xi_i)\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}|\rho_i(X(n))|\widetilde{u}\rangle\xi_i^{-n-1}d\xi_i\Big) \\ &= -\langle\widetilde{\Psi}|\rho_i(X[f])|\widetilde{u}\rangle \\ &= \langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle_\eta.\end{aligned}$$

This proves Claim 3. □

Now we are ready to prove Theorem 3.4.1. Put

$$|\widetilde{u}\rangle = |u\otimes X_1(-1)0\otimes\cdots\otimes X_M(-1)0\rangle.$$

The above argument shows that

$$\langle\widetilde{\Psi}|\widetilde{u}\rangle d\eta_1\dots d\eta_M = \langle\widetilde{\Psi}|u\otimes X_1(-1)0\otimes\cdots\otimes X_M(-1)0\rangle d\eta_1\dots d\eta_M$$

is regarded as an element of $T^*_{P_1}C\otimes\cdots\otimes T^*_{P_M}C$, if $P_k \neq Q_j$ and $P_j \neq P_k$, $j\neq k$, and depends meromorphically on P_k. Hence, by Hartog's theorem, it defines an element of $H^0\big(C^M, \omega_C^{\boxtimes}\big(\sum_{i<j}*\Delta_{ij} + \sum_{i=1}^M\sum_{j=1}^N *\pi_i^{-1}(Q_j)\big)\big)$. We denote this meromorphic section by

$$\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|u\rangle dP_1\,dP_2\dots dP_M.$$

The assertions (0) and (1) are clear by our definition. For the assertion (2) note that the meromorphic form defined above from the data

$$\widetilde{\mathfrak{X}} = (C; Q_1,\dots,Q_N,P_1,\dots,P_M;\eta_1,\dots,\eta_{N+M})$$

and the data

$$\widetilde{\mathfrak{X}}_\sigma = (C; Q_1,\dots,Q_N,P_{\sigma(1)},\dots,P_{\sigma(M)};\eta_1,\dots,\eta_N,\eta_{N+\sigma(1)},\dots,\eta_{N+\sigma(M)})$$

are the same. This implies assertion (2).

Assertion (3) follows from Claim 2.

Let us prove assertion (4). Let the point P' be in a small neighbourhood U of the point P with local coordinate z with center P. Let us choose a meromorphic function $f \in H^0\big(C, \mathcal{O}_C\big(*\big(P + \sum_{k=1}^{N+M}Q_k\big)\big)\big)$ such that

$$f = z^{-1} + \text{regular at } P.$$

Moreover, changing the local coordinate at P if necessary, we may assume that $f = z^{-1}$. Then $w = z - z(P')$ is a local coordinate of C at P'. As a cotangent vector at each point of $(P, P') \times C \times \cdots \times C$ (M copies of C),

$$F = \langle \Psi | X(P) Y(P') X_1(P_1) \dots X_M(P_M) | \Phi \rangle dP dP'$$

is equal to

$$\langle \widehat{\Psi}^* | X(-1)0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle dz dw,$$

where

$$\langle \widehat{\Psi}^* | = \iota(\langle \Psi |), \quad \iota \colon \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_{M+2}}(\widehat{\mathfrak{X}})$$

and

$$|\widetilde{\Phi}\rangle = |\Phi \otimes X_1(-1)0_{P_1} \otimes \cdots \otimes X_M(-1)0_{P_M}\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0}_M}.$$

Then we have

$$\begin{aligned} \langle \widehat{\Psi}^* | X(-1)0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle = & - \langle \widehat{\Psi} | (X[f]) Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle \\ & - \sum_{k=1}^{N+M} \langle \widehat{\Psi} | Y(-1)0_{P'} \otimes \rho_k(X[f]) | \widetilde{\Phi} \rangle. \end{aligned} \tag{3.4.1}$$

The second term of the right hand side of (3.4.1) is written as

$$- \sum_{k=1}^{N+M} \langle \widetilde{\Psi} | Y(P') | \rho_k(X[f]) \widetilde{\Phi} \rangle dP'.$$

Hence, it is holomorphic at the point P'. On the other hand, putting $a = z(P')$ we have

$$\begin{aligned} (X[f])Y(-1)|0_{P'}\rangle &= \left(X\left[\tfrac{1}{w+a}\right]\right)(Y[w^{-1}])|0_{P'}\rangle \\ &= \left(\frac{[X,Y]}{a}[w^{-1}] - \frac{\ell \cdot (X,Y)}{a^2}\right)|0_{P'}\rangle. \end{aligned}$$

Hence the first term of the right hand side of (3.4.1) has the form

$$\begin{aligned} & \frac{\ell \cdot (X,Y)}{a^2} \langle \Psi | X_1(P_1) \dots X_M(P_M) | \Phi \rangle \\ & \qquad - \frac{1}{a} \langle \Psi | [X,Y](P') X_1(P_1) \dots X_M(P_M) | \Phi \rangle. \end{aligned}$$

Since $-a = z(P) - z(P')$, we have the desired result.

A similar argument proves assertion (5). □

Remark 3.4.2. Using the formal coordinates ξ_j, we let $(\xi^{'}_{j_1}, \xi^{''}_{j_2}, \xi^{(3)}_{j_3}, \dots, \xi^{(n)}_{j_n})$ be a system of formal coordinates of $C \times C \times \cdots \times C$ center at $(Q_{j_1}, Q_{j_2}, \dots, Q_{j_n})$, where $\xi^{(m)}_j$ is a copy of ξ_j. The correlation function

$$\langle \Psi | X_1(z_1) X_2(z_2) \dots X_n(z_n) | \Phi \rangle dz_1 dz_2 \dots dz_n$$

has the Laurent expansion

$$\begin{aligned} & \sum_{m_i \in \mathbb{Z}} \langle \Psi | \rho_1(X_1(m_1)) \rho_2(X_2(m_2)) \\ & \qquad \dots \rho_n(X_n(m_n)) \Phi \rangle \xi^{'-m_1-1}_{j_1} \xi^{''-m_2-1}_{j_2} \dots \xi^{(n)-m_n-1}_{j_n} d\xi^{'}_{j_1} \xi^{''}_{j_2} \dots d\xi^{(n)}_{j_n} \end{aligned}$$

at the point $(Q_{j_1}, Q_{j_2}, \dots, Q_{j_n})$.

Furthermore we can show the following Theorem.

Theorem 3.4.3.

(1) *Put*

$$\begin{aligned}
&\langle\Psi|T(z)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz^2 dP_1 dP_2\cdots dP_M\\
&\quad=\frac{1}{2(g^*+l)}\lim_{w\to z}\left\{\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(z)J^a(w)X_1(P_1)\right.\\
&\qquad\qquad X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz dw dP_1 dP_2\cdots dP_M\\
&\qquad\left.-\frac{\ell\dim\mathfrak{g}}{(z-w)^2}\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz dw dP_1 dP_2\cdots dP_M\right\}.
\end{aligned}$$

Then, this is well-defined and for $k=1,\dots,N$, we have

$$\begin{aligned}
&\operatorname*{Res}_{\xi_k=0}(\xi_k^{n+1}\langle\Psi|T(\xi_k)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle d\xi_k) dP_1 dP_2\cdots dP_M\\
&\qquad=\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|\rho_k(L_n)\Phi\rangle dP_1 dP_2\cdots dP_M,
\end{aligned}$$

where $\{J^1,\dots,J^{\dim\mathfrak{g}}\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$. (See (2.2.1).) Thus we have an expansion

$$\begin{aligned}
&\langle\Psi|T(\xi_k)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle d\xi_k^2 dP_1 dP_2\cdots dP_M\\
&\qquad=\sum_{n\in\mathbb{Z}}\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|\rho_k(L_n)\Phi\rangle\xi_k^{-n-2}d\xi_k^2 dP_1 dP_2\cdots dP_M.
\end{aligned}$$

(2) *For a holomorphic coordinate transformation $w=w(z)$ we have*

$$\begin{aligned}
&\langle\Psi|T(w)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dw^2 dP_1 dP_2\cdots dP_M\\
&\quad=\langle\Psi|T(z)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz^2 dP_1 dP_2\cdots dP_M\\
&\qquad-\frac{c_v}{12}\{w(z);z\}\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz^2 dP_1 dP_2\cdots dP_M,
\end{aligned}$$

where $\{w(z);z\}$ is the Schwarzian derivative.

3.5. The Riemann Sphere $\mathbb{P}^1$.

In this section we study the spaces of vacua when the underlying curve is the Riemann sphere $\mathbb{P}^1$. The details can be found in [36].) We regard $\mathbb{P}^1$ as $\mathbb{C}\cup\{\infty\}$ and let z be a global coordinate of $\mathbb{C}$. For a positive integer $N\geqslant 1$ let us choose N-points $z_1,z_2,\dots z_N$ of $\mathbb{P}^1$ and put

$$\xi_j=\begin{cases} z-z_j & \text{if } z_j\neq\infty\\ 1/z & \text{if } z_j=\infty.\end{cases}$$

Then $\mathfrak{X}=(\mathbb{P}^1;z_1,\dots,z_N;\xi_1,\dots,\xi_N)$ is an N-pointed curve with formal coordinates. Choose $\vec{\lambda}=(\lambda_1,\dots\lambda_N)\in(P_\ell)^N$ and put

$$V_{\vec{\lambda}}=V_{\lambda_1}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}V_{\lambda_N}.$$

Proposition 3.5.1. *The natural restriction mapping*

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}},\mathbb{C})\longrightarrow\operatorname{Hom}_{\mathbb{C}}(V_{\vec{\lambda}},\mathbb{C}).$$

induces an injective homomorphism

$$j\colon\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})\hookrightarrow\operatorname{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}},\mathbb{C}).$$

Moreover, if none of z_i is ∞, then an element $\langle\psi| \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$ is in the image of j, if and only if for each $k = 1, 2, \ldots, N$ and for any element

$$|\phi_k\rangle = |v_1 \otimes \cdots \otimes v_{k-1} \otimes \lambda_k \otimes v_{k+1} \otimes \cdots \otimes v_N\rangle \in V_{\vec{\lambda}},$$

we have the equalities

$$\sum_{\substack{\vec{m_k} \\ |\vec{m_k}|=\ell_k}} \prod_{j\neq k} \binom{\ell_k}{m_j} (z_j - z_k)^{-m_j} \left\langle \psi \right| \prod_{i\neq k} \rho_i(X_\theta)^{m_i} \left| \phi_k \right\rangle = 0, \tag{3.5.1}$$

where

$$\ell_k = \ell - (\theta, \lambda_k) + 1,$$
$$\vec{m_k} = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_N), \quad \mathbb{Z} \ni m_i \geqslant 0,$$
$$|\vec{m_k}| = \sum_{i\neq k} m_i,$$

and $|\lambda_k\rangle$ is a highest weight vector of V_{λ_k}.

Note that the equality (3.5.1) is equivalent to

$$\langle \Psi | \rho_k(X_\theta(-1))^{\ell_k} |\phi\rangle = 0,$$

if we put $\langle\psi| = j(\langle\Psi|)$, $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. (See (3.5.2) below.)

Proof. By the gauge condition, for $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and $X \in \mathfrak{g}$ we have

$$\sum_{j=1}^{N} \langle \Psi | \rho_j(X \otimes 1) = 0.$$

This implies that the image of j lies in $\mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$.

Now let us show that (3.5.1) is the necessary condition for $\langle\psi|$ to be in $j(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}))$. For $\langle\psi| = j(\langle\Psi|)$ and $h = 1/(z - z_k)$, by an argument similar to the one in the proof of Theorem 3.4.1, we have the following equalities:

$$\begin{aligned} &\langle \Psi | \rho_k(X_\theta(-1))^{\ell_k} |\phi\rangle \\ &\quad = \langle \Psi | \rho_k(X_\theta \otimes h)^{\ell_k} |\phi\rangle \\ &\quad = (-1)^{\ell_k} \sum_{\substack{\vec{m_k} \\ |\vec{m_k}|=\ell_k}} \binom{\ell_k}{m_j} (z_j - z_k)^{-m_k} \left\langle \Psi \prod_{j\neq k} \rho_j(X_\theta)^{m_j} \right| \left. \phi \right\rangle. \end{aligned} \tag{3.5.2}$$

Moreover, by Theorem 2.1.4, we have

$$X_\theta(-1)^{\ell_k} |\lambda_k\rangle = 0.$$

Hence the equalities (3.5.2) imply the equality (3.5.1).

Next we show that j is injective. Assume $j(\langle\Psi|) = 0$. By induction on p we prove that the restriction $\langle\Psi|\big|_{F_p(\mathcal{H}_{\vec{\lambda}})}$ of $\langle\Psi|$ to $F_p\mathcal{H}_{\vec{\lambda}}$ is zero. For $p = 0$ this is true by the assumption $j(\langle\Psi|) = 0$. Assume $\langle\Psi|\big|_{F_p(\mathcal{H}_{\vec{\lambda}})} = 0$. An element $|\phi\rangle \in F_{p+1}\mathcal{H}_{\vec{\lambda}}$ can be written as

$$\rho_j(X(-n))|v\rangle, \quad |v\rangle \in F_p\mathcal{H}_{\vec{\lambda}},$$

for some j, a positive integer n and $X \in \mathfrak{g}$. Put $f = 1/(z - z_j)^n$. Then, we have

$$\begin{aligned}\langle \Psi|\phi\rangle &= \langle \Psi|\rho_j(X(-n))|v\rangle \\ &= \langle \Psi|\rho_j(X \otimes f)|v\rangle \\ &= -\sum_{i\neq j}\langle \Psi|\rho_i(X \otimes f)|v\rangle = 0,\end{aligned}$$

since f is holomorphic at z_i, $i \neq j$, hence $\rho_i(X \otimes f)|v\rangle \in F_q\mathcal{H}_{\vec{\lambda}}$. This shows $\langle \Psi| = 0$.

Finally let us show the sufficiency of the condition (3.5.1). First we show that each element $\langle \psi| \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$ can be extended to an element

$$\langle \Psi^{(p)}| \in \mathrm{Hom}_{\mathbb{C}}(F_p\mathcal{M}_{\vec{\lambda}}, \mathbb{C}),$$

where

$$\begin{aligned}\mathcal{M}_{\vec{\lambda}} &= \mathcal{M}_{\lambda_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{M}_{\lambda_N}, \\ F_p\mathcal{M}_{\vec{\lambda}} &= \bigoplus_{p_1+\cdots+p_N=p} F_{p_1}\mathcal{M}_{\lambda_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} F_{p_N}\mathcal{M}_{\lambda_N}\end{aligned}$$

and $\mathcal{M}_{\lambda_j}$ is the Verma module with highest weight λ_j (2.1.14). More precisely, by induction on p, we prove the existence of an extension

$$\langle \Psi^{(p)}| \in \mathrm{Hom}(F_p\mathcal{M}_{\vec{\lambda}}, \mathbb{C}),$$

such that if we have

$$\rho_j(X \otimes f)|v\rangle \in F_p\mathcal{M}_{\vec{\lambda}}, \quad j = 1, 2, \ldots, N,$$

for an element $f \in H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(*\sum_{j=1}^N z_j\right)\right)$ and $|v\rangle \in F_p\mathcal{M}_{\vec{\lambda}}$, then we have

$$\sum_{j=1}^N \langle \Psi^{(p)}|\rho_j(X \otimes f)|v\rangle = 0. \tag{3.5.5}$$

For $p = 0$, put

$$\langle \Psi^{(0)}| = \langle \psi|.$$

Then the equality (3.5.5) holds for a constant function f. Now assume that we can define $\langle \Psi^{(q)}|$. An element $|\phi\rangle$ of $F_{q+1}M(\vec{\lambda})$ is expressed as

$$\rho_j(-n)|v\rangle,$$

for suitable j, a positive integer n and $|v\rangle \in F_q\mathcal{M}_{\vec{\lambda}}$. Put

$$h(z) = \frac{1}{(z - z_j)^n}$$

and define $\langle \Psi^{(q+1)}|$ by

$$\langle \Psi^{(q+1)}|\phi\rangle = -\sum_{i\neq j}\langle \Psi^{(q)}|\rho_i(X \otimes h)|v\rangle. \tag{3.5.6}$$

Since $h(z)$ is holomorphic at z_i, $i \neq j$, $\rho_i(X \otimes h)|v\rangle \in F_q\mathcal{M}_{\vec{\lambda}}$. Therefore, the right hand side of (3.5.6) is well-defined. The equality (3.5.5) is a consequence of the definition (3.5.6) and the induction hypothesis. Thus we have

$$\langle \Psi| \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_{\vec{\lambda}}, \mathbb{C}),$$

which satisfies the gauge condition

$$\sum_{j=1}^{N}\langle\Psi|\rho_j(X\otimes f)=0, \tag{3.5.7}$$

for all $f\in H^0\big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}\big(*\sum z_j\big)\big)$.

Now we show that $\langle\Psi|$ actually defines an element $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}},\mathbb{C})$, hence by the gauge condition (3.5.7), is an element of $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$.

It is enough to show that for each k, $1\leqslant k\leqslant N$, we have

$$\langle\Psi|(\mathcal{M}_{\lambda_1}\otimes\cdots\otimes\mathcal{M}_{\lambda_{k-1}}\otimes\mathcal{J}_{\lambda_k}\otimes\mathcal{M}_{\lambda_{k+1}}\otimes\cdots\otimes\mathcal{M}_{\lambda_N})=0. \tag{3.5.8}$$

Note that $\mathcal{J}_{\lambda_k}$ is defined to be $U(\widehat{\mathfrak{p}}_-)|J_{\lambda_k}\rangle$, where $|J_{\lambda_k}\rangle=X_\theta(-1)^{\ell-(\theta,\lambda_k)+1}|\lambda_k\rangle$ and is the maximal proper $\widehat{\mathfrak{g}}$ submodule of $\mathcal{M}_{\lambda_k}$ such that $\mathcal{H}_{\lambda_k}=\mathcal{M}_{\lambda_k}/\mathcal{J}_{\lambda_k}$. First we show

$$\langle\Psi|(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_{k-1}}\otimes\mathcal{J}_{\lambda_k}\otimes V_{\lambda_{k+1}}\otimes\cdots\otimes V_{\lambda_N})=0, \tag{3.5.9}$$

by induction on the degree of the filtration $F_\bullet\mathcal{J}_{\lambda_k}:=F_\bullet\mathcal{M}_{\lambda_k}\cap\mathcal{J}_{\lambda_k}$. By our assumption

$$\langle\Psi|(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_{k-1}}\otimes J_{\lambda_k}\otimes V_{\lambda_{k+1}}\otimes\cdots\otimes V_{\lambda_N})=0.$$

Assume that $\langle\Psi|$ is zero on $F_p\mathcal{J}_{\lambda_k}$. Then, an element of $F_{p+1}\mathcal{J}_{\lambda_k}$ is a $\mathbb{C}$-linear combination of elements

$$|v_1\otimes\cdots\otimes v_{k-1}\otimes Y(-n)w\otimes v_{k+1}\otimes\cdots\otimes v_N\rangle,$$

where $v_i\in V_{\lambda_i}$, $Y\in\mathfrak{g}$, $w\in F_p\mathcal{J}_{\lambda_k}$ and n is a positive integer. Since $\frac{1}{(\zeta-z_k)^n}$ is holomorphic at z_i, $i\neq k$, we have

$$Y\otimes\frac{1}{(\zeta-z_k)^n}|v_i\rangle\in V_{\lambda_i}. \tag{3.5.10}$$

Hence, by (3.5.7), (3.5.10) and the induction hypothesis we have

$$\begin{aligned}
&\langle\Psi|v_1\otimes\cdots\otimes v_{k-1}\otimes Y(-n)w\otimes v_{k+1}\otimes\cdots\otimes v_N\rangle\\
&\quad=\left\langle\Psi\,\middle|\,\rho_k\left(Y\otimes\tfrac{1}{(\zeta-z_k)^n}\right)v_1\otimes\cdots\otimes v_{k-1}\otimes w\otimes v_{k+1}\otimes\cdots\otimes v_N\right\rangle\\
&\quad=-\sum_{i\neq k}\left\langle\Psi\,\middle|\,\rho_i\left(Y\otimes\tfrac{1}{(\zeta-z_k)^n}\right)v_1\otimes\cdots\otimes v_{k-1}\otimes w\otimes v_{k+1}\otimes\cdots\otimes v_N\right\rangle\\
&\quad=0.
\end{aligned}$$

Hence we obtain the desired result (3.5.8).

Next consider

$$\langle\Psi|Y(-n)v_1\otimes\cdots\otimes v_{k-1}\otimes\mathcal{J}_{\lambda_k}\otimes v_{k+1}\otimes\cdots\otimes v_N\rangle,$$

for $Y\in\mathfrak{g}$, $|v_i\rangle\in V_{\lambda_i}$ and a positive integer n. As we have

$$Y\otimes\frac{1}{(\zeta-z_k)^n}|v_i\rangle\in V_{\lambda_i},\quad Y\otimes\frac{1}{(\zeta-z_k)^n}\mathcal{J}_{\lambda_k}\subset\mathcal{J}_{\lambda_k},$$

for $i\geqslant 2$, we have

$$\begin{aligned}
&\langle\Psi|Y(-n)v_1\otimes\cdots\otimes v_{k-1}\otimes\mathcal{J}_{\lambda_k}\otimes v_{k+1}\otimes\cdots\otimes v_N\rangle\\
&\quad=-\sum_{i=2}^{N}\left\langle\Psi\,\middle|\,\rho_i\left(Y\otimes\tfrac{1}{(\zeta-z_k)^n}\right)v_1\otimes\cdots\otimes v_{k-1}\otimes\mathcal{J}_{\lambda_k}\otimes v_{k+1}\otimes\cdots\otimes v_N\right\rangle\\
&\quad=0.
\end{aligned}$$

In this way by induction on the degree of the filtration $F_\bullet(\mathcal{M}_{\lambda_1}\otimes\cdots\otimes\mathcal{M}_{\lambda_{k-1}}\otimes\mathcal{M}_{\lambda_{k+1}}\otimes\cdots\otimes\mathcal{M}_{\lambda_N})$ we can show that (3.5.8) holds. □

Corollary 3.5.2.

(1) *We have*

$$\mathcal{V}_{\lambda,0,0}(\mathfrak{X}) = 0.$$

(2) *We have*

$$\mathcal{V}_{\lambda,\mu,0}(\mathfrak{X}) = \begin{cases} \mathbb{C}, & \text{for } \mu = \lambda^\dagger \\ 0, & \text{otherwise} \end{cases}$$

(3) *For* $\vec{\lambda} = (\mu, \nu, \lambda)$ *we have*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \simeq W_{\vec{\lambda}},$$

where

$$W_{\vec{\lambda}} = \{\, \psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}) \mid \text{condition } (*) \,\}$$

and the condition $(*)$ *is given as follows. Let*

$$\mathfrak{k}_\theta = \mathbb{C}X_\theta \oplus \mathbb{C}X_{-\theta} \oplus \mathbb{C}[X_\theta, X_{-\theta}]$$

be the principal three-dimensional subalgebra of $\mathfrak{g}$ *and let*

$$V_\lambda = \bigoplus_{i=0}^{\ell/2} W_{\lambda,i}$$

be the irreducible decomposition as a $\mathfrak{k}_\theta$*-module. Then the condition* $(*)$ *is*

$$\psi|_{W_{\mu,h} \otimes W_{\nu,i} \otimes W_{\lambda,j}} = 0, \quad \text{if } h + i + j > \ell. \tag{$*$}$$

Let us consider correlation functions. Since there are no non-zero holomorphic one-forms on $\mathbb{P}^1$, we get for $X \in \mathfrak{g}$ and $|v\rangle \in V_{\vec{\lambda}}$ by Theorem 3.4.1, (5), that

$$\langle \Psi | X(z) | v \rangle dz = \sum_{j=1}^{N} \frac{1}{z - z_j} \langle \Psi | \rho_j(X) v \rangle dz. \tag{3.5.11}$$

Note that if one of the z_j, say z_1 is the point at infinity ∞, then the term

$$\frac{1}{z - z_j} \langle \Psi | \rho_1(X) v \rangle dz$$

in (3.5.11) disappears. This is because the residue at the point at infinity ∞ of the form

$$\sum_{j=2}^{N} \frac{1}{z - z_j} \langle \Psi | \rho_j(X) v \rangle dz$$

is

$$-\sum_{j=2}^{N} \langle \Psi | \rho_j(X) v \rangle = \langle \Psi | \rho_1(X) v \rangle$$

by the gauge condition.

Similarly for X, $Y \in \mathfrak{g}$ we have

$$\begin{aligned} &\langle \Psi | X(z) Y(w) | v \rangle dz dw \\ &\quad = \frac{\ell \cdot (X, Y)}{(z - w)^2} \langle \Psi | v \rangle dz dw + \frac{1}{z - w} \langle \Psi | [X, Y](w) | v \rangle dz dw \\ &\qquad + \sum_{j=1}^{N} \frac{1}{w - z_j} \langle \Psi | Y(w) | \rho_j(X) v \rangle dz dw, \end{aligned} \tag{3.5.12}$$

since there are no non-zero holomorphic two-forms on $\mathbb{P}^1 \times \mathbb{P}^1$. By (3.5.11) the equality (3.5.12) is rewritten in the form

$$\begin{aligned}
&\langle \Psi | X(z)Y(w)|v\rangle dzdw \\
&\quad = \frac{\ell \cdot (X,Y)}{(z-w)^2}\langle \Psi|v\rangle dzdw + \frac{1}{z-w}\sum_{j=1}^{N}\frac{1}{z-z_j}\langle \Psi|\rho_j([X,Y])v\rangle dzdw \\
&\qquad + \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{(z-z_j)(w-z_k)}\langle \Psi|\rho_j(X)\rho_k(Y)v\rangle dzdw. \qquad (3.5.13)
\end{aligned}$$

This can also be expressed in the following form.

$$\begin{aligned}
&\langle \Psi | X(z)Y(w)|v\rangle dzdw \\
&\quad = \frac{\ell \cdot (X,Y)}{(z-w)^2}\langle \Psi|v\rangle dzdw + \frac{1}{w-z}\sum_{j=1}^{N}\frac{1}{w-z_j}\langle \Psi|\rho_j([Y,X])v\rangle dzdw \\
&\qquad + \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{(z-z_k)(w-z_j)}\langle \Psi|\rho_j(Y)\rho_k(X)v\rangle dzdw. \qquad (3.5.14)
\end{aligned}$$

Here, again, if $z_1 = \infty$, then the terms containing z_1 should be omitted. By (3.5.14) and Theorem 3.4.3, for the correlation function of the energy momentum tensor we have the following expression.

$$\begin{aligned}
&\langle \Psi | T(z)|v\rangle (dz)^2 \qquad (3.5.15)\\
&\quad = \frac{1}{2(g^*+\ell)}\sum_{j,k}\frac{1}{(z-z_j)(z-z_k)}\cdot\sum_{a=1}^{\dim \mathfrak{g}}\langle \Psi|\rho_j(J^a)\rho_k(J^a)|v\rangle (dz)^2,
\end{aligned}$$

for $\langle \Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ and $|v\rangle \in V_{\vec{\lambda}}$. Put

$$\Omega_{jk} = \sum_{a=1}^{\dim \mathfrak{g}}\rho_j(J^a)\rho_k(J^a) \qquad (3.5.16)$$

and

$$\langle \psi| = j(\langle \Psi|) \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}).$$

Then we have the following expression which will be used below.

$$\langle \Psi | T(z)|v\rangle (dz)^2 = \frac{1}{2(g^*+\ell)}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\langle \psi|\Omega_{jk}|v\rangle}{(z-z_j)(z-z_k)}(dz)^2. \qquad (3.5.17)$$

If $z_1 = \infty$ this formula should read

$$\langle \Psi | T(z)|v\rangle (dz)^2 = \frac{1}{2(g^*+\ell)}\sum_{j=2}^{N}\sum_{k=2}^{N}\frac{\langle \psi|\Omega_{jk}|v\rangle}{(z-z_j)(z-z_k)}(dz)^2. \qquad (3.5.18)$$

3.6. Elliptic Curves.

Let E be an elliptic curve with period matrix $(1,\tau)$ with $\tau \in H$, where H is the upper half plane:

$$E = \mathbb{C}/(1,\tau).$$

Let us consider a one-pointed curve $\mathfrak{X} = (E;[0];z)$ of genus 1 with a formal coordinate where $[0]$ is the origin of the elliptic curve E and z is a global coordinate of $\mathbb{C}$. The space of vacua $\mathcal{V}_\lambda^\dagger(\mathfrak{X})$ is given by the conditions

$$\begin{aligned}\langle\Psi|X\otimes\wp^{(n)}(z) &= 0 \quad n = 0,1,\dots,\\ \langle\Psi|X\otimes 1 &= 0,\end{aligned}$$

where $\wp(z)$ is the Weierstrass $\wp$ function. For an element $\langle\Psi| \in \mathcal{V}_\lambda^\dagger(\mathfrak{X})$, an element $X \in \mathfrak{g}$ and an element $|v\rangle \in V_\lambda$, the one-form $\langle\Psi|X(z)|v\rangle dz$ has the expansion

$$\begin{aligned}&\langle\Psi|X(z)|v\rangle dz\\ &\qquad = \sum_{n\in\mathbb{Z}}\langle\Psi|X(n)v\rangle z^{-n-1}dz\\ &\qquad = \sum_{n=0}^{\infty}\langle\Psi|X(-n)v\rangle z^{n-1}dz.\end{aligned}$$

Since there is no meromorphic one form on the elliptic curve E which has a pole of order one at the origin and holomorphic outside the origin, the form $\langle\Psi|X(z)|v\rangle dz$ is holomorphic. Therefore, we have

$$\langle\Psi|X(z)|v\rangle dz = \langle\Psi|X(-1)|v\rangle dz \tag{3.6.1}$$

and if $n \neq 1$ we have

$$\langle\Psi|X(-n)|v\rangle = 0. \tag{3.6.2}$$

More generally, for an element $\langle\Psi| \in \mathcal{V}_\lambda^\dagger(\mathfrak{X})$, an element $X \in \mathfrak{g}$ and an element $|\Phi\rangle \in \mathcal{H}_\lambda$, the one-form $\langle\Psi|X(z)|\Phi\rangle dz$ has the expansion

$$\begin{aligned}&\langle\Psi|X(z)|\Phi\rangle dz\\ &\qquad = \sum_{n=1}^{n_0}\langle\Psi|X(-n)\Phi\rangle\frac{(-1)^{n-1}}{(n-1)!}\wp^{(n-1)}(z)dz\\ &\qquad\quad + \Big(\langle\Psi|X(-1)\Phi\rangle - \sum_{n=1}^{n_0}\alpha_n\langle\Psi|X(n)\Phi\rangle\Big)dz,\end{aligned}$$

where

$$\alpha_n = \lim_{z\mapsto 0}\left(\frac{(-1)^{n-1}}{(n-1)!}\wp^{(n-1)}(z) - \frac{1}{z^n}\right).$$

For example, we have

$$\begin{aligned}\alpha_1 = 0, \qquad \alpha_3 &= 3\sum_{(m,n)\neq(0,0)}\frac{1}{(m+n\tau)^4},\\ \alpha_5 &= 5\sum_{(m,n)\neq(0,0)}\frac{1}{(m+n\tau)^6}\end{aligned}$$

and $\alpha_{2n} = 0$. Similarly, using Theorem 3.4.1, (4), we can prove the following lemma.

Lemma 3.6.1. *For an element* $\langle\Psi| \in \mathcal{V}_\lambda^\dagger(\mathfrak{X})$, $X, Y \in \mathfrak{g}$ *and an element* $|v\rangle \in V_\lambda$, *we have*

$$\begin{aligned}\langle\Psi|X(z)Y(w)|v\rangle dzdw &= \langle\Psi|[X,Y](-1)v\rangle(\zeta(z-w)-\zeta(z)+\zeta(w))dzdw \\ &\quad + \langle\Psi|X(-1)Y(-1)v\rangle dzdw,\end{aligned}$$

where $\zeta(z)$ *is the Weierstrass* ζ *function.*

Proof. Since we have

$$\langle\Psi|X = 0, \quad X \in \mathfrak{g},$$

we have

$$\langle\Psi|[X,Y](-1)v\rangle = \langle\Psi|X(-1)Yv\rangle = -\langle\Psi|Y(-1)Xv\rangle,$$

where we use the fact that

$$X(-1)Y - YX(-1) = [X,Y](-1) = XY(-1) - Y(-1)X.$$

By Theorem 3.4.1 the meromorphic form $\langle\Psi|X(z)Y(w)|v\rangle dzdw$ on $E \times E$ has an expansion in a neighbourhood of the origin of $E \times E$

$$\begin{aligned}&\left\{\frac{\langle\Psi|[X,Y](-1)v\rangle}{z-w} + \frac{\langle\Psi|Y(-1)Xv\rangle}{z}\right. \\ &\qquad \left. + \frac{\langle\Psi|X(-1)Yv\rangle}{w} + \text{holomorphic}\right\} dzdw \\ &= \left\{\frac{\langle\Psi|[X,Y](-1)v\rangle}{z-w} - \frac{\langle\Psi|[X,Y](-1)v\rangle}{z}\right. \\ &\qquad \left. + \frac{\langle\Psi|[X,Y](-1)v\rangle}{w} + \text{holomorphic}\right\} dzdw.\end{aligned}$$

Since the constant term of the expansion is $\langle\Psi|X(-1)Y(-1)v\rangle dzdw$ we obtained the desired result. □

By Theorem 3.4.3, (1) we also have the following lemma.

Lemma 3.6.2. *For* $|v\rangle \in V_\lambda$ *we have*

$$\langle\Psi|T(z)|v\rangle dz^2 = \frac{1}{2(g^*+\ell)} \sum_{a=1}^{\dim\mathfrak{g}} \langle\Psi|J^a(-1)J^a(-1)|v\rangle dz^2.$$

Again by Theorem 3.4.3, (1) we also have

$$\begin{aligned}\langle\Psi|T(z)|v\rangle dz^2 &= \sum_{n\in\mathbb{Z}} \langle\Psi|L_n v\rangle z^{-n-2} dz^2 \\ &= \langle\Psi|L_0 v\rangle \wp(z) dz^2 + \langle\Psi|L_{-2}v\rangle dz^2 + \cdots \\ &= \Delta_\lambda \langle\Psi|v\rangle \wp(z) dz^2 + \langle\Psi|L_{-2}v\rangle dz^2 + \cdots \\ &= \langle\Psi|L_{-2}v\rangle dz^2.\end{aligned}$$

In particular, we have

$$\langle\Psi|L_{-1}v\rangle = 0.$$

4. The Sheaf of Conformal Blocks

4.1. The Sheaf of Vacua Attached to a Family of Stable Curves.

Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N; \eta_1, \dots, \eta_N)$ be a family of N-pointed stable curves of genus g with formal coordinates. We assume that $\mathcal{B}$ is a finite dimensional complex manifold and that each fiber of the family $\mathfrak{F}$ satisfies the condition (Q) in section 1.1, but we do *not* assume that the family is connected. The main purpose of the present section is to define the sheaf of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ attached to the family and prove that it is a coherent $\mathcal{O}_{\mathcal{B}}$-module.

Definition 4.1.1. The *sheaf* $\widehat{\mathfrak{g}}_N(\mathcal{B})$ *of affine Lie algebras over* $\mathcal{B}$ is the sheaf of $\mathcal{O}_{\mathcal{B}}$-modules

$$\widehat{\mathfrak{g}}_N(\mathcal{B}) = \mathfrak{g} \otimes_{\mathbb{C}} \Big(\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j)) \Big) \oplus \mathcal{O}_{\mathcal{B}} \cdot c$$

with the following commutation relation, which is $\mathcal{O}_{\mathcal{B}}$-bilinear,

$$\begin{aligned} &[(X_1 \otimes f_1, \dots, X_N \otimes f_N), (Y_1 \otimes g_1, \dots, Y_N \otimes g_N)] \\ &\quad = ([X_1, Y_1] \otimes (f_1 g_1), \dots, [X_N, Y_N] \otimes (f_N g_N)) \\ &\qquad \oplus c \cdot \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0} (g_j df_j) \end{aligned}$$

where $c \in \text{Center}$, $X_j, Y_j \in \mathfrak{g}$, $f_j, g_j \in \mathcal{O}_{\mathcal{B}}((\xi_j))$.

Put

$$\widehat{\mathfrak{g}}(\mathfrak{F}) = \mathfrak{g} \otimes_{\mathbb{C}} \pi_*(\mathcal{O}_{\mathcal{C}}(*S)), \tag{4.1.1}$$

where we define

$$S = \sum_{j=1}^{N} s_j(\mathcal{B})$$

$$\pi_*(\mathcal{O}_{\mathcal{C}}(*S)) = \varinjlim_{k} \pi_*(\mathcal{O}_{\mathcal{C}}(kS)).$$

There is a sheaf version of the homomorphism defined in (1.1.6), by using the formal neighbourhoods η_j:

$$\tilde{t} : \pi_*(\mathcal{O}_{\mathcal{B}}(*S)) \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))$$

and we may regard $\widehat{\mathfrak{g}}(\mathfrak{F})$ as a Lie subalgebra of $\widehat{\mathfrak{g}}_N(\mathcal{B})$.

Fix a non-negative integer ℓ. For any $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, we define

$$\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) = \mathcal{O}_{\mathcal{B}} \otimes_{\mathbb{C}} \mathcal{H}_{\vec{\lambda}}, \tag{4.1.2}$$

$$\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) = \underline{Hom}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{H}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}). \tag{4.1.3}$$

The pairing (2.2.13) induces an $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle \cdot | \cdot \rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \times \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \longrightarrow \mathcal{O}_{\mathcal{B}}, \tag{4.1.4}$$

which is perfect with respect to the filtration introduced below. The sheaf of affine Lie algebra $\widehat{\mathfrak{g}}_N(\mathcal{B})$ acts on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ by

$$\begin{aligned} &\Big(\Big(X_1 \otimes \sum_{n \in \mathbb{Z}} a_n^{(1)} {\xi_1}^n \Big), \dots, \Big(X_N \otimes \sum_{n \in \mathbb{Z}} a_n^{(N)} {\xi_N}^n \Big) \Big) (F \otimes |\Psi\rangle) \\ &\qquad = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} (a_n^{(j)} F) \otimes \rho_j(X_j(n)) |\Psi\rangle. \end{aligned} \tag{4.1.5}$$

The action of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ is the dual action of the one on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, that is,

$$\langle \Psi a | \Phi \rangle = \langle \Psi | a \Phi \rangle, \quad \text{for any } a \in \widehat{\mathfrak{g}}_N. \tag{4.1.6}$$

Definition 4.1.2. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) = \mathcal{H}_{\vec{\lambda}}(\mathcal{B})/\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}),$$
$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) = \underline{Hom}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_{\mathcal{B}}).$$

These are sheaves of $\mathcal{O}_{\mathcal{B}}$-modules on $\mathcal{B}$. The sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is called the *sheaf of vacua* attached to the family $\mathfrak{F}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is called the *sheaf of covacua*. Note that we have

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) = \{\, \langle\Psi| \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \mid \langle\Psi|a = 0, \quad \text{for any } a \in \widehat{\mathfrak{g}}(\mathfrak{F}) \,\}.$$

The pairing (4.1.4) induces a non-degenerate $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle\cdot|\cdot\rangle : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \longrightarrow \mathcal{O}_{\mathcal{B}}. \tag{4.1.7}$$

Lemma 4.1.3. *For a point $s \in \mathcal{B}$ we put*

$$\mathfrak{F}_s := (\pi^{-1}(s); s_1(s), \ldots, s_N(s); \eta_1|_{\pi^{-1}(s)}, \ldots, \eta_N|_{\pi^{-1}(s)}),$$
$$\mathbb{C}_s := \mathcal{O}_{\mathcal{B},s}/\mathfrak{m}_s$$

where $\mathfrak{m}_s$ is the maximal ideal of the stalk $\mathcal{O}_{\mathcal{B},s}$. Then, we have the following canonical isomorphisms

$$\mathbb{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}_{\vec{\lambda}},$$
$$\mathbb{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathfrak{g}}_N(\mathcal{B}) \simeq \widehat{\mathfrak{g}}_N,$$
$$\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathbb{C}_s \simeq \widehat{\mathfrak{g}}(\mathfrak{F}_s),$$
$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathbb{C}_s \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s).$$

More generally, for a holomorphic mapping $f : Y \to \mathcal{B}$ we let $\mathfrak{F}_Y$ be the pull-back of the family $\mathfrak{F}$ by the morphism f. Then, we have the following canonical isomorphisms

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}_{\vec{\lambda}}(Y),$$
$$\widehat{\mathfrak{g}}_N(\mathcal{B}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y \simeq \widehat{\mathfrak{g}}_N(Y),$$
$$\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y \simeq \widehat{\mathfrak{g}}(\mathfrak{F}_Y),$$
$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_Y).$$

Moreover, the actions of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ and $\widehat{\mathfrak{g}}_N(Y)$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$, $\mathcal{H}_{\vec{\lambda}}(Y)$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}(Y)$ defined in (4.1.5) *and* (4.1.6) *and the action of $\widehat{\mathfrak{g}}_N$ on $\mathcal{H}_{\vec{\lambda}}$ are compatible with respect to the above canonical isomorphisms.*

Proof. The first, second and the fifth isomorphisms are clear from the definition. If k is sufficiently large, we always have the base change

$$\mathbb{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \pi_*(\mathcal{O}_{\mathcal{C}}(kS)) \simeq H^0\Big(C_s, \mathcal{O}_{C_s}\Big(k\sum_{j=1}^{N} s_j(s)\Big)\Big),$$

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \pi_*(\mathcal{O}_{\mathcal{C}}(kS)) \simeq H^0\Big(C_Y, \mathcal{O}_{C_Y}\Big(k\sum_{j=1}^{N} s_j(Y)\Big)\Big),$$

since we have

$$H^1\Big(C_s, \mathcal{O}_{C_s}\Big(k\sum_{j=1}^{N} s_j(s)\Big)\Big) = 0$$

for each point $s \in \mathcal{B}$. (See for example, [17, Chap. 3, Corollary 12.9] or [3, Chap. 3 Corollary 3.5].) This implies the third isomorphism.

Finally let us consider the following commutative diagram of exact sequences

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
\mathbb{C}_s \otimes (\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) & \xrightarrow{\beta} & (\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes \mathbb{C}_s)(\mathbb{C}_s \otimes \mathcal{H}_{\vec{\lambda}}(\mathcal{B})) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \epsilon} \\
\mathbb{C}_s \otimes \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) & \xrightarrow{\gamma} & \mathcal{H}_{\vec{\lambda}} \\
\downarrow & & \downarrow \\
\mathbb{C}_s \otimes \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) & \xrightarrow{\delta} & \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The above argument shows that the mapping β is surjective and the mapping γ an isomorphism. Hence, the commutativity of the diagram implies that the mapping γ induces isomorphism between the $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\epsilon)$. Therefore, the mapping δ is an isomorphism. $\square$

Next we shall prove the coherence of the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. First we introduce filtrations $\{F_\bullet\}$ which are generalizations of the ones given in section 3.1. The filtration $\{F_\bullet\}$ on $\mathcal{O}_{\mathcal{B}}((\xi_j))$ is defined by

$$F_p\mathcal{O}_{\mathcal{B}}((\xi_j)) = \mathcal{O}_{\mathcal{B}}[[\xi_j]]\xi_j^{-p}, \quad p \in \mathbb{Z}.$$

The filtration $\{F_\bullet\}$ on $\widehat{\mathfrak{g}}_N(\mathcal{B})$ is defined by

$$F_p\widehat{\mathfrak{g}}_N(\mathcal{B}) = \begin{cases} \mathfrak{g} \otimes \bigoplus_{j=1}^{N} F_p\mathcal{O}_{\mathcal{B}}((\xi_j)) \oplus \mathcal{O}_{\mathcal{B}} \cdot c & p \geqslant 0 \\ \mathfrak{g} \otimes \bigoplus_{j=1}^{N} F_p\mathcal{O}_{\mathcal{B}}((\xi_j)) & p < 0. \end{cases} \tag{4.1.7}$$

The filtration $\{F_\bullet\}$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is defined by

$$F_p\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) = \mathcal{O}_{\mathcal{B}} \otimes F_p\mathcal{H}_{\vec{\lambda}} \tag{4.1.8}$$

where $F_p\mathcal{H}_{\vec{\lambda}} = \sum_{d \leqslant p} \mathcal{H}_{\vec{\lambda}}(d)$ and

$$\mathcal{H}_{\vec{\lambda}}(d) = \sum_{d_1+\cdots+d_N=d} \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N).$$

It is easy to see that

$$F_p\widehat{\mathfrak{g}}_N(\mathcal{B}) \cdot F_q\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \subseteq F_{p+q}\mathcal{H}_{\vec{\lambda}}(\mathcal{B}). \tag{4.1.9}$$

We have the following isomorphism

$$Gr_\bullet^F\widehat{\mathfrak{g}}_N \simeq \mathcal{O}_{\mathcal{B}} \otimes_{\mathbb{C}} \bigoplus_{j=1}^{N} \mathfrak{g} \otimes \mathbb{C}[\xi_j, \xi_j^{-1}] \oplus \mathcal{O}_{\mathcal{B}}c,$$

$$Gr_\bullet^F\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{O}_{\mathcal{B}} \otimes_{\mathbb{C}} \mathcal{H}_{\vec{\lambda}},$$

where $\deg \xi_j = -1$, $\deg c = 0$, and the degree d part of $\mathcal{H}_{\vec{\lambda}}$ is $\mathcal{H}_{\vec{\lambda}}(d)$.

On $\widehat{\mathfrak{g}}(\mathfrak{F})$ and on $\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ we introduce the induced filtrations from those of $\widehat{\mathfrak{g}}_N$ and $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ respectively. On $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ we introduce the quotient filtrations from that of $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$. Then we have the following exact sequences of graded $\mathcal{O}_{\mathcal{B}}$-modules.

$$0 \longrightarrow Gr_\bullet^F \widehat{\mathfrak{g}}(\mathfrak{F}) \longrightarrow Gr_\bullet^F \widehat{\mathfrak{g}}_N(\mathcal{B}), \tag{4.1.10}$$

$$0 \longrightarrow Gr_\bullet^F (\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) \longrightarrow Gr_\bullet^F \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \longrightarrow Gr_\bullet^F \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \longrightarrow 0.$$

We consider the graded Lie subalgebra $\mathcal{O}_{\mathcal{B}} \otimes (\mathfrak{g} \otimes \bigoplus_{j=1}^N \mathbb{C}[\xi_j^{-1}]\xi_j^{-M})$ of $Gr_\bullet^F \widehat{\mathfrak{g}}_N(\mathcal{B})$, for a positive integer M.

The Riemann-Roch Theorem implies the following lemma.

Lemma 4.1.4. *There exists a positive integer M such that*

$$\mathcal{O}_{\mathcal{B}} \otimes \Big(\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{j=1}^N \mathbb{C}[\xi_j^{-1}]\xi_j^{-M}\Big) \hookrightarrow Gr_\bullet^F \widehat{\mathfrak{g}}(\mathfrak{F}). \tag{4.1.11}$$

Fix a positive integer M satisfying (4.1.11) and define a graded $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{V}$ by

$$\mathcal{V} = Gr_\bullet^F \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \Big/ \Big(\mathcal{O}_{\mathcal{B}} \otimes \Big(\bigoplus_{j=1}^N \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\xi_j^{-1}]\xi_j^{-M}\Big)\Big) Gr_\bullet^F \mathcal{H}_{\vec{\lambda}}\,. \tag{4.1.12}$$

On the other hand, by (4.1.9) and the proof of Lemma 3.1.9, we have

$$F_p(\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) = \sum_{p_1+p_2=p} F_{p_1}(\widehat{\mathfrak{g}}(\mathfrak{F})) \cdot F_{p_2}(\mathcal{H}_{\vec{\lambda}}(\mathcal{B})).$$

Hence we have

$$Gr_\bullet^F (\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) = Gr_\bullet^F \widehat{\mathfrak{g}}(\mathfrak{F}) \cdot Gr_\bullet^F \mathcal{H}_{\vec{\lambda}}(\mathcal{B}).$$

Therefore, by Lemma 4.1.4 we have the following lemma.

Lemma 4.1.5. *There is a surjective $\mathcal{O}_{\mathcal{B}}$-module homomorphism*

$$\mathcal{V} \longrightarrow Gr_\bullet^F \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \longrightarrow 0.$$

Put

$$V = \mathcal{H}_{\vec{\lambda}} \Big/ \Big(\mathfrak{g} \otimes \bigoplus_{j=1}^N \mathbb{C}[\xi_j^{-1}]\xi_j^{-M}\Big)\mathcal{H}_{\vec{\lambda}}\,.$$

Then, as $\mathcal{O}_{\mathcal{B}}$-modules we have

$$\mathcal{V} \simeq \mathcal{O}_{\mathcal{B}} \otimes V.$$

By Proposition 3.1.6, V is of finite dimension. Hence, $\mathcal{V}$ is a coherent $\mathcal{O}_{\mathcal{B}}$-module. Therefore, by Lemma 4.1.5, $Gr_\bullet^F \mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a coherent $\mathcal{O}_{\mathcal{B}}$-module. Hence, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is also a coherent $\mathcal{O}_{\mathcal{B}}$-module. Thus, we have proved the following theorem.

Theorem 4.1.6. *The sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F})$ are coherent $\mathcal{O}_{\mathcal{B}}$-modules.*

From Theorem 3.2.4 we infer the following theorem.

Theorem 4.1.7. *If two families $\mathfrak{F}$ and $\widetilde{\mathfrak{F}}$ of N-pointed stable curves with formal neighbourhoods induce the same family of N-pointed stable curves with first order neighbourhoods, then there are canonical isomorphisms*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}(\widetilde{\mathfrak{F}})$$
$$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}^\dagger(\widetilde{\mathfrak{F}})$$

as $\mathcal{O}_{\mathcal{B}}$-modules.

4.2. Local Freeness I (the Smooth Case).

In this section we shall prove local freeness of the sheaf of vacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. For that purpose we first introduce a certain $\mathcal{O}_{\mathcal{B}}$-submodule $\mathcal{L}(\mathfrak{F})$ of

$$\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j^{-1}))\frac{d}{d\xi_j}$$

and an action of $\mathcal{L}(\mathfrak{F})$ on the sheaves of vacua and covacua as first order twisted differential operators. In section 5.1 this action will be used to define a projectively flat connection on the sheaf of vacua.

Let $\mathfrak{F}^{(0)} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N)$ be a *versal family* of N-pointed stable curves of genus g. We let Σ be the locus of double points of the fibers of $\mathfrak{F}^{(0)}$ and D be $\pi(\Sigma)$. Note that Σ is a non-singular submanifold of codimension two in $\mathcal{C}$ and D is a divisor in $\mathcal{B}$ whose irreducible components D_i, $i = 1, 2, \ldots, k$ are non-singular. (See Lemma 1.2.10.) Assume that formal coordinates

$$\eta_j\colon \widehat{\mathcal{O}}_{\mathcal{C}/s_j(\mathcal{B})} \simeq \mathcal{O}_{\mathcal{B}}[[\xi]], \qquad j = 1, 2, \ldots, N$$

are given. For simplicity, in the following we assume that $\eta_j^{-1}(\xi)$ is holomorphic in a neighbourhood of $s_j(\mathcal{B})$. The general case can be treated by approximating formal coordinates by holomorphic ones. We use the following notation freely

$$S_j = s_j(\mathcal{B}), \qquad S = \sum_{j=1}^{N} S_j, \qquad \xi_j = \eta_j^{-1}(\xi).$$

Then, $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ is a family of N-pointed stable curves of genus g with formal neighbourhoods. For each $\vec{\lambda} \in (P_\ell)^{\oplus N}$ we can define the sheaf of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ and the sheaf of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. These are respectively a subsheaf of $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ and a quotient sheaf of $\mathcal{H}_{\vec{\lambda}}(\mathfrak{F})$.

First recall that we have an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-S)(\Theta_{\mathcal{C}/\mathcal{B}}(mS)) \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-S) \longrightarrow \bigoplus_{j=1}^{N}\bigoplus_{k=0}^{m} \mathcal{O}_{\mathcal{B}}\xi_j^{-k}\frac{d}{d\xi_j} \longrightarrow 0,$$

for any positive integer m. From this exact sequence we obtain the exact sequence

$$0 \longrightarrow \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(mS)) \xrightarrow{b_m} \bigoplus_{j=1}^{N}\bigoplus_{k=0}^{m} \mathcal{O}_{\mathcal{B}}\xi_j^{-k}\frac{d}{d\xi_j} \xrightarrow{\vartheta_m} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \longrightarrow 0. \tag{4.2.1}$$

Hence, we have the following exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules (see (1.3.17))

$$0 \longrightarrow \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \xrightarrow{b} \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}[\xi_j^{-1}]\frac{d}{d\xi_j} \xrightarrow{\vartheta} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \longrightarrow 0. \tag{4.2.2}$$

Note that the mappings b and b_m correspond to the Laurent expansions with respect to ξ_j up to zero'th order. To define the first order twisted differential operators acting on the sheaves of vacua and covacua, we need to modify the exact sequence (4.2.1) in the following way.

There is an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \Theta_{\mathcal{C}} \xrightarrow{d\pi} \pi^*\Theta_{\mathcal{B}} \longrightarrow 0,$$

where $\Theta_{\mathcal{C}/\mathcal{B}}$ is a sheaf of vector fields tangent to the fibers of π. Put

$$\Theta'_{\mathcal{C},\pi} = d\pi^{-1}(\pi^{-1}\Theta_{\mathcal{B}}(-\log D)).$$

Hence, $\Theta'_{\mathcal{C},\pi}$ is a sheaf of vector field on $\mathcal{C}$ tangent to Σ whose horizontal components are constant along the fibers of π. That is, $\Theta'_{\mathcal{C},\pi}$ consists of germs of holomorphic vector fields of the form

$$a(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^{m'} b_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m'+1}^{n} b_i(u)\frac{\partial}{\partial u_i},$$

where $(z, u_1, \ldots, u_n)$ is a system of local coordinates such that the mapping π is expressed as the projection

$$\pi(z, u_1, \ldots, u_n) = (u_1, \ldots, u_n)$$

and $\pi(\Sigma) = D$ is given by the equation

$$u_1 \cdot u_2 \cdots u_{m'} = 0.$$

More generally, we can define a sheaf $\Theta'_{\mathcal{C}}(mS)_\pi$ as the one consisting of germs of meromorphic vector fields of the form

$$A(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^{m'} B_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m'+1}^{n} B_i(u)\frac{\partial}{\partial u_i},$$

where $A(z,u)$ has the poles of order at most m along S. Now we have an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}(mS) \longrightarrow \Theta'_{\mathcal{C}}(mS)_\pi \xrightarrow{d\pi} \pi^{-1}\Theta_{\mathcal{B}}(-\log D) \longrightarrow 0. \tag{4.2.3}$$

Note that $\Theta'_{\mathcal{C}}(mS)_\pi$ has the structure of a sheaf of Lie algebras by the usual bracket operation of vector fields and the above exact sequence is the one of sheaves of Lie algebras.

For $m > \frac{1}{N}(2g-2)$, by (4.2.3) we have and exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules.

$$0 \longrightarrow \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(mS) \longrightarrow \pi_*\Theta'_{\mathcal{C}}(mS)_\pi \xrightarrow{d\pi} \Theta_{\mathcal{B}}(-\log D) \longrightarrow 0, \tag{4.2.4}$$

which is also an exact sequence of sheave of Lie algebras. Taking $m \to \infty$ we obtain the exact sequence

$$0 \longrightarrow \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \longrightarrow \pi_*\Theta'_{\mathcal{C}}(*S)_\pi \xrightarrow{d\pi} \Theta_{\mathcal{B}}(-\log D) \longrightarrow 0. \tag{4.2.5}$$

The exact sequences (4.2.1) and (4.2.3) are related by the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \longrightarrow & \pi_*\Theta'_{\mathcal{C}}(*S)_\pi & \xrightarrow{d\pi} & \Theta_{\mathcal{B}}(-\log D) & \longrightarrow & 0 \\
 & & \| & & \downarrow p & & \downarrow \rho & & \\
0 & \longrightarrow & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \longrightarrow & \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}[\xi_j^{-1}]\frac{d}{d\xi_j} & \xrightarrow{\vartheta} & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) & \longrightarrow & 0,
\end{array}$$

where ρ is the Kodaira-Spencer mapping of the family $\mathfrak{F}^{(0)}$ and p is given by taking the non-positive part of the $\frac{d}{d\xi_j}$ part of the Laurent expansions of the vector fields in $\pi_*\Theta_{\mathcal{C}}(mS)_\pi$ at $s_j(\mathcal{B})$. Since our family $\mathfrak{F}^{(0)}$ is versal, the Kodaira-Spencer mapping ρ is an isomorphism of $\mathcal{O}_{\mathcal{B}}$-modules. Therefore, p is an isomorphism. Let

$$\tilde{p}\colon \pi_*\Theta'_{\mathcal{C}}(*S)_\pi \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))\frac{d}{d\xi_j}$$

be obtained by taking the $\frac{d}{d\xi_j}$ part of the Laurent expansions at $s_j(\mathcal{B})$. Since p is an isomorphism, $\widetilde{p}$ is injective. Put

$$\mathcal{L}(\mathfrak{F}) := \widetilde{p}(\pi_*\Theta'_{\mathcal{C}}(*S)_\pi).$$

Then, we have the exact sequence.

$$0 \longrightarrow \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \longrightarrow \mathcal{L}(\mathfrak{F}) \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log D) \longrightarrow 0 \tag{4.2.6}$$

of $\mathcal{O}_{\mathcal{B}}$-modules. The Lie bracket $[\ ,\ ]_d$ of $\mathcal{L}(\mathfrak{F})$ is obtained from the bracket on $\pi_*\Theta'_{\mathcal{C}}(*S)_\pi$ by the mapping $\widetilde{p}$. Thus, for $\vec{\ell}$, $\vec{m} \in \mathcal{L}(\mathfrak{F})$ we have

$$[\vec{\ell}, \vec{m}]_d = [\vec{\ell}, \vec{m}]_0 + \theta(\vec{\ell})(\vec{m}) - \theta(\vec{m})(\vec{\ell}) \tag{4.2.7}$$

where $[\cdot, \cdot]_0$ is the usual bracket of formal vector fields and the action of $\theta(\vec{\ell})$ on

$$\vec{m} = \left(m_1 \frac{d}{d\xi_1}, \ldots, m_N \frac{d}{d\xi_N}\right)$$

is defined by

$$\left(\theta(\vec{\ell})(m_1)\frac{d}{d\xi_1}, \ldots, \theta(\vec{\ell})(m_N)\frac{d}{d\xi_N}\right).$$

Then, the exact sequence (4.2.6) is also an exact sequence of sheaves of Lie algebras.

Let us define an action of $\mathcal{L}(\mathfrak{F})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

Definition 4.2.1. For $\vec{\ell} = (\underline{l}_1, \ldots, \underline{l}_N) \in \mathcal{L}(\mathfrak{F})$, the action $D(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is defined by

$$D(\vec{\ell})(F \otimes |\Phi\rangle) = \theta(\vec{\ell})(F) \otimes |\Phi\rangle - F \cdot \left(\sum_{j=1}^{N} \rho_j(T[\underline{l}_j])\right)|\Phi\rangle, \tag{4.2.8}$$

where

$$F \in \mathcal{O}_{\mathcal{B}}, \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}.$$

The following proposition is an easy consequence of Lemma 2.2.8. and the definition.

Proposition 4.2.2. *The action $D(\vec{\ell})$ of $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ defined above has the following properties.*

(1) *For any $f \in \mathcal{O}_{\mathcal{B}}$ we have*

$$D(f\vec{\ell}) = fD(\vec{\ell}).$$

(2) *For $\vec{\ell}$, $\vec{m} \in \mathcal{L}(\mathfrak{F})$ we have*

$$[D(\vec{\ell}), D(\vec{m})] = D([\vec{\ell}, \vec{m}]_d) + \frac{c_v}{12}\sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}\left(\frac{d^3\ell_j}{d\xi_j^3} m_j d\xi_j\right) \cdot id.$$

(3) *For $f \in \mathcal{O}_{\mathcal{B}}$ and $|\phi\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ we have*

$$D(\vec{\ell})(f|\phi\rangle) = (\theta(\vec{\ell})(f))|\phi\rangle + fD(\vec{\ell})|\phi\rangle.$$

Namely, $D(\vec{\ell})$ is a first order twisted differential operator, if $\theta(\vec{\ell}) \neq 0$.

We define the dual action of $\mathcal{L}(\mathfrak{F})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ by

$$D(\vec{\ell})(F\otimes\langle\Psi|)=(\theta(\vec{\ell})F)\otimes\langle\Psi|+\sum_{j=1}^{N}F\cdot(\langle\Psi|\rho_j(T[\underline{l}_j]). \tag{4.2.9}$$

where

$$F\in\mathcal{O}_{\mathcal{B}},\qquad \langle\Psi|\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}).$$

Then, for any $|\widetilde{\Phi}\rangle\in\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ and $\langle\widetilde{\Psi}|\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$, we have

$$\{D(\vec{\ell})\langle\widetilde{\Psi}|\}|\widetilde{\Phi}\rangle+\langle\widetilde{\Psi}|\{D(\vec{\ell})|\widetilde{\Phi}\rangle\}=\theta(\vec{\ell})\langle\widetilde{\Psi}|\widetilde{\Phi}\rangle. \tag{4.2.10}$$

This agrees with the usual definition of the dual connection. See also Proposition 4.2.8 and (4.2.14) below.

Now we shall show that the operator $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$.

Proposition 4.2.3. *For any $\vec{\ell}\in\mathcal{L}(\mathfrak{F})$ we have*

$$D(\vec{\ell})(\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}))\subset\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}).$$

Hence, $D(\vec{\ell})$ operates on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. Moreover, it is a first order differential operator, if $\theta(\vec{\ell})\neq 0$.

Proof. An element of $\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is a linear combination of elements of the form

$$F\otimes\Big(\sum_{j=1}^{N}\rho_j(X\otimes t_j(h))|\Phi\rangle\Big),$$

where

$$F\in\mathcal{O}_{\mathcal{B}},\quad X\in\mathfrak{g},\quad h\in\pi_*\mathcal{O}_{\mathcal{C}}(*S),\quad |\Phi\rangle\in\mathcal{H}_{\vec{\lambda}}$$

and $t_j(h)$ is the Laurent expansion of h at $S_j=s_j(\mathcal{B})$ with respect to the parameter ξ_j. First we shall show the following equality as operators on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

$$\Big[D(\vec{\ell}),\sum_{j=1}^{N}\rho_j(X\otimes t_j(h))\Big]=\sum_{j=1}^{N}\rho_j\left(X\otimes\left\{\theta(\vec{\ell})(t_j(h))+\underline{\ell}_j(t_j(h))\right\}\right) \tag{4.2.11}$$

where $\theta(\vec{\ell})$ operates on the coefficients of $t_j(h)$. By Proposition 4.2.2, (3) it is enough to show the equality (4.2.11) as operators on $\mathcal{H}_{\vec{\lambda}}$. For $|\Phi\rangle\in\mathcal{H}_{\vec{\lambda}}$, by (4.2.8) we have

$$\begin{aligned}
&D(\vec{\ell})\Big(\sum_{j=1}^{N}\rho_j(X\otimes t_j(h))|\Phi\rangle\Big)-\sum_{j=1}^{N}\rho_j(X\otimes t_j(h))(D(\vec{\ell})|\Phi\rangle)\\
&\qquad=\sum_{j=1}^{N}\{\rho_j(X\otimes\theta(\vec{\ell})(t_j(h))-T[\underline{\ell}_j]\rho_j(X\otimes t_j(h))\}|\Phi\rangle\\
&\qquad\quad+\sum_{j=1}^{N}\rho_j(X\otimes t_j(h))T[\underline{\ell}_j]|\Phi\rangle\\
&\qquad=\sum_{j=1}^{N}\{\rho_j(X\otimes\theta(\vec{\ell})(t_j(h))+\rho_j(X\otimes\underline{\ell}_j(t_j(h)))\}|\Phi\rangle.
\end{aligned}$$

But $\theta(\vec{\ell})(t_j(h))+\underline{\ell}_j(t_j(h))$ is nothing but a Laurent expansion at $s_j(\mathcal{B})$ of a meromorphic function $\tau(h)$ where $\tau=\widetilde{p}^{-1}(\vec{\ell})\in\pi_*(\Theta'_{\mathcal{C}}(*S)_\pi)$. Hence, we have the desired result. □

Similarly we can show that $D(\vec{\ell})$ acts on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$. Another proof of this fact can be found in the proof of Proposition 4.2.8 below.

Corollary 4.2.4. *The $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free on $\mathcal{B} \setminus D$.*

Proof. By Lemma 4.1.3, for any point $x \in \mathcal{B} \setminus D$ we have an isomorphism

$$\mathbb{C}_x \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x),$$

where we put

$$\mathfrak{F}_x := (C_x = \pi^{-1}(x), s_1(x), \ldots, s_N(x); \eta_1|_{C_x}, \ldots, \eta_N|_{C_x}).$$

Let $v_1, \ldots, v_m$ be local holomorphic sections of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ in a neighbourhood of x such that $\{v_1(x), \ldots, v_m(x)\}$ is a basis of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x)$. Suppose that there is a non-trivial relation

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \tag{4.2.12}$$

where a_i's are holomorphic function in a neighbourhood of x. By our assumption

$$a_1(x) = a_2(x) = \cdots = a_m(x) = 0.$$

Changing the order of suffices if necessary, we may assume that there is a positive integer k such that

$$\begin{aligned} &a_1 \in \mathfrak{m}_x^k \setminus \mathfrak{m}_x^{k+1}, \\ &a_i \in \mathfrak{m}_x^l, \quad l \geqslant k, \quad i = 2, 3, \ldots, m. \end{aligned}$$

We choose a_i's in such a way that the positive number k is the smallest among the relations (4.2.12). Let τ be a nowhere vanishing local holomorphic vector field in a neighbourhood of x such that $\tau(a_1) \in \mathfrak{m}_x^{k-1}$. There exists $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}) = \tau$. Applying $\tau = \theta(\vec{\ell})$ to the equality (4.2.12) we obtain the equality

$$\sum_{i=1}^{m} \Big(\tau(a_i) + \sum_{j=1}^{m} a_j \alpha_{ji} \Big) v_i = 0, \tag{4.2.13}$$

where

$$D(\vec{\ell})(v_i) = \sum_{j=1}^{m} \alpha_{ij} v_j.$$

Then, the relation (4.2.13) is non trivial and

$$\tau(a_1) + \sum_{j=1}^{m} \alpha_{j1} \in \mathfrak{m}_x^{k-1}.$$

This contradicts our assumption. Therefore, $v_1, \ldots, v_m$ are $\mathcal{O}_{\mathcal{B}}$-linearly independent. Hence, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module at x. $\square$

For a coherent $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{G}$, the locus M consisting of points at which $\mathcal{G}$ is not locally free, is a closed analytic subset of $\mathcal{B}$ of codimension at least 2. Therefore, we have the following corollary.

Corollary 4.2.5. *Let W be the maximal subset of $\mathcal{B}$ over which $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free. Then, W is an analytic subset of $\mathcal{B}$ and*

$$W \subsetneqq D.$$

Since we defined

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}) = \underline{Hom}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_{\mathcal{B}}),$$

we have the following corollary.

Corollary 4.2.6. *$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})|_{\mathcal{B}\setminus D}$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module and for any subvariety Y of $\mathcal{B}\setminus D$ we have an $\mathcal{O}_Y$-module isomorphism*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}|_Y).$$

These two corollaries play a crucial role in proving locally freeness in general. The above corollaries imply the following theorem.

Theorem 4.2.7. *If $\mathfrak{F}$ is a family of N-pointed smooth curves with formal neighbourhoods such that the induced family $\mathfrak{F}^{(0)}$ of N-pointed smooth curves is not necessarily versal, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ are locally free $\mathcal{O}_{\mathcal{B}}$-modules and they are dual to each other.*

Another important consequence of the above discussions is the following.

Proposition 4.2.8. *For each element $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$, $D(\vec{\ell})$ acts on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$. Moreover, if $\theta(\vec{\ell}) \neq 0$, then $D(\vec{\ell})$ acts on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ as a twisted first order differential operator.*

Proof. Choose

$$\langle\widetilde{\Psi}| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}).$$

For any element $X \in \mathfrak{g}$, $f \in \pi_*\mathcal{O}_{\mathcal{C}}(*S)$ and $|\widetilde{\Phi}\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, by Proposition 4.2.3 and (4.2.10) we have

$$\{D(\vec{\ell})\langle\widetilde{\Psi}|\}(X\otimes f|\widetilde{\Phi}\rangle = \theta(\vec{\ell})(\langle\widetilde{\Psi}|X\otimes f|\widetilde{\Phi}\rangle - \langle\widetilde{\Psi}|\{D(\vec{\ell})(X\otimes f|\widetilde{\Phi}\rangle)\} = 0.$$

Thus we conclude

$$D(\vec{\ell})\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}).$$

The remaining statement is an easy consequence of definition (4.2.9). □

Note that for the bilinear pairing $\langle\cdot|\cdot\rangle\colon \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \longrightarrow \mathcal{O}_{\mathcal{B}}$ given in (4.1.7), we have the equality

$$\{D(\vec{\ell})\langle\Psi|\}|\Phi\rangle + \langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = \theta(\vec{\ell})(\langle\Psi|\Phi\rangle). \tag{4.2.14}$$

4.3. The Family of One Pointed Stable Curves of Genus 1.

Since the arguments in section 4.4 and 5.3 below are complicated, it seems apropriate first to consider a simple non-trivial case which illustrates our arguments.

First let us construct a versal family of stable curves of genus one.

Let z and w be coordinates of $\mathbb{P}^1$ with center 0 and ∞, respectively with

$$zw = 1.$$

Put

$$D = \{\, q \in \mathbb{C} \mid |q| < 1 \,\}, \quad X = \{\, R \in \mathbb{P}^1 \mid |z(R)| < 1 \,\},$$
$$Y = \{\, P \in \mathbb{P}^1 \mid |w(P)| < 1 \,\}.$$

and

$$S = \{\, (x,y,q) \in \mathbb{C}^3 \mid xy = q,\ |x| < 1,\ |y| < 1,\ |q| < 1 \,\},$$
$$Z = \left\{\, (R,q) \in \mathbb{P}^1 \times D \mid R \in \mathbb{P}^1 \setminus \{X \cup Y\} \text{ or } R \in X \text{ and } |z(R)| > |q| \,\right\},$$
$$W = \left\{\, (P,q) \in \mathbb{P}^1 \times D \mid P \in \mathbb{P}^1 \setminus \{X \cup Y\} \text{ or } P \in X \text{ and } |w(P)| > |q| \,\right\}.$$

Let us introduce an equivalence relation on $S \sqcup Z \sqcup W$. The equivalence relation is generated by the following relations:

(1) A point $(R,q) \in X \times D \cap Z$ and a point $(x,y,q') \in S$ are equivalent if

$$(x,y,q') = \left(z(R), \frac{q}{z(R)}, q\right).$$

(2) A point $(P,q) \in Y \times D \cap W$ and a point $(x,y,q') \in S$ are equivalent if

$$(x,y,q') = \left(\frac{q}{w(P)}, w(P), q\right).$$

(3) A point $(R,q) \in Z$ and a point $(P,q') \in W$ are equivalent if

$$q' = q, \quad z(R)w(P) = q.$$

Put

$$\mathcal{E} = (S \cup Z \cup W)/\sim .$$

Then $\mathcal{E}$ is a two-dimensional complex manifold. For points $(x,y,q) \in S$, $(R,q) \in Z$ and $(P,q) \in W$ we denote the corresponding points in $\mathcal{E}$ by $[(x,y,q)]$, $[(R,q)]$ and $[(P,q)]$, respectively. We have a natural holomorphic mapping

$$\pi\colon \mathcal{E} \longrightarrow D.$$

For $q \in D$ put

$$\begin{aligned} S(q) &= \{\, (x,y) \in \mathbb{C}^2 \mid xy = q \,\}, \\ Z(q) &= \{\mathbb{P}^1 - X \cup Y\} \cup \{\, R \in X \mid |z(R)| > |q| \,\}, \\ W(q) &= \{\mathbb{P}^1 - X \cup Y\} \cup \{\, P \in Y \mid |w(P)| > |q| \,\}. \end{aligned}$$

By reasons which will become clear later we use mainly the coordinate w so that

$$\mathbb{P}^1 = \{\, (w:1) \mid w \in \mathbb{C}\} \cup \{(1:0)\,\}.$$

Assume that $q \neq 0$. We can rewrite

$$\begin{aligned} \mathbb{P}^1 - X \cup Y &= \{\, (w:1) \mid |w| = 1 \,\}, \\ Z(q) &= \{\, (w:1) \mid 1 \leqslant |w| < 1/|q| \,\}, \\ W(q) &= \{\, (w:1) \mid |q| < |w| \leqslant 1 \,\}. \end{aligned}$$

The above equivalence relation induces the one on $S(q) \cup Z(q) \cup W(q)$ which can be expressed as

- $(w:1) \in X \times q \cap Z(q)$ (i.e. $1 \leq |w| < 1/|q|$) and $(x,y,q) \in S(q)$ are equivalent if $(x,y,q) = (1/w, qw, q)$.
- $(w':1) \in Y \times q \cap W(q)$ (i.e. $1 \leq |w| < 1/|q|$) and $(x,y,q) \in S(q)$ are equivalent if $(x,y,q) = (q/w', w', q)$.

Note that if $q \neq 0$, then any $(x,y,q) \in S(q)$ is either equivalent to the point $(w:1) \in X \times q \cap Z(q)$ (when $|x| \leqslant 1$) or to the point $(w':1) \in Y \times q \cap W(q)$ (when $|y| \leqslant 1$). Therefore, the quotient manifold $\mathcal{E}(q) = (S(q) \cup Z(q) \cup W(q))/\sim$ is obtained by identifying $1 \leqslant |w| < 1/|q|$ and $|q| < |w'| \leqslant 1$ by $w' = qw$. In other word, $\mathcal{E}(q)$ is the quotient manifold $E_{\langle q \rangle} = \mathbb{C}^*/\langle q \rangle$ where $\langle q \rangle$ is the infinite cyclic group of analytic automorphism generated by

$$w \mapsto qw.$$

Hence, $\mathcal{E}(q)$ is the elliptic curve $E_{\langle q \rangle}$, if $q \neq 0$ and a rational curve with an ordinary double point for $q = 0$. Note that $\mathcal{E}(q)$ is the fiber $\pi^{-1}(q)$ of $\pi\colon \mathcal{E} \to D$ and that the point $[1]$ of $\mathcal{E}(q)$ corresponding to $w = 1$ is the origin of the elliptic curve. Moreover, if $q = 0$, the fiber $\pi^{-1}(0)$ of π is a rational curve with an ordinary double point obtained by identifying the points $\infty = (0:1)$ and $0 = (1:0)$.

It is easy to show that $\frac{dw}{w}$ is a holomorphic one-form on $\mathcal{E}(q)$ for $q \neq 0$, and the family $\pi\colon \mathcal{E} \to D$ is versal at each point $q \in D$.

The mapping

$$\begin{aligned} D &\longrightarrow \mathcal{E}, \\ q &\longmapsto [(q, 1, q)], \end{aligned}$$

defines a holomorphic section o of the family $\pi\colon \mathcal{E} \to D$.

Let us consider the family $\mathfrak{F} = (\mathcal{E} \to D; o; u)$ with $u = w - 1$ of one-pointed stable curves of genus one with coordinate. Let us consider the sheaf of vacua $\mathcal{V}_0(\mathfrak{F})$. First we will construct a formal sections of $\mathcal{V}_0^\dagger(\mathfrak{F})$ starting from the data on the fiber $\mathcal{E}(0)$ at the origin.

Let

$$\nu\colon \widetilde{\mathcal{E}}(0) = \mathbb{P}^1 \longrightarrow \mathcal{E}(0)$$

be the normalization of $\mathcal{E}(0)$. Then, we have

$$\begin{aligned} &\nu(0) = \nu(\infty) = \text{the double point}, \\ &\nu(1) = o(0). \end{aligned}$$

We know that $\mathcal{V}_0^\dagger((\mathcal{E}(0); o(0), u))$ is canonically isomorphic to

$$\bigoplus \mathcal{V}^\dagger_{\lambda^\dagger,\lambda,0}((\mathbb{P}^1; \infty, 0, 1; \eta, \xi, \xi - 1)).$$

For simplicity, put

$$V^\dagger_{\lambda^\dagger,\lambda,0} = \mathcal{V}^\dagger_{\lambda^\dagger,\lambda,0}((\mathbb{P}^1; \infty, 1, 0; \eta, \xi - 1, \xi)).$$

The vector space $V^\dagger_{\lambda^\dagger,\lambda,0}$ is one-dimensional and there is the element $\langle \Psi | \in V^\dagger_{\lambda^\dagger,\lambda,0}$ such that

$$\langle \Psi | v^\dagger \otimes v \otimes 0 \rangle = (v^\dagger | v)$$

for any $|v^\dagger\rangle \in \mathcal{H}_{\lambda^\dagger}$ and $|v\rangle \in \mathcal{H}_\lambda$, where $|0\rangle$ is the highest weight vector of $\mathcal{H}_0$ and $(\cdot|\cdot)$ is the bilinear pairing

$$\mathcal{H}_{\lambda^\dagger} \otimes \mathcal{H}_\lambda \longrightarrow \mathbb{C}$$

defined in section 2.2. Let us construct an element

$$\langle \Psi^{(\lambda)} | \in \mathcal{V}_0^\dagger(\mathfrak{F})$$

from $\langle \Psi |$.

For any non-negative integer d we let $\{v^k(d)\}$ and $\{v_k(d)\}$, $k = 1, 2, \ldots, m_d$ be bases of $\mathcal{H}_{\lambda^\dagger,d}$ and $\mathcal{H}_{\lambda,d}$ such that

$$(v^j(d) | v_k(d)) = \delta^j_k.$$

For each element $|\phi\rangle \in \mathcal{H}_0$ $\langle \Psi_d | \in \mathcal{H}_0^\dagger$ is defined by

$$\langle \Psi_d | \phi \rangle = \sum_{i=1}^{m_d} \langle \Psi | v^i(d) \otimes 0 \otimes v_i(d) \otimes \phi \rangle.$$

Now $\langle \Psi^{(\lambda)} |$ is defined by

$$\langle \Psi^{(\lambda)} | \phi \rangle = \sum_{d=0}^{\infty} \langle \Psi_d | \phi \rangle q^d.$$

This is a formal power series in q and from our construction it is not clear that it converges. Later we shall show that $q^{\Delta_\lambda}\langle\Psi^{(\lambda)}|\phi\rangle$ is a formal solution of a differential equation of Fuchsian type so that it converges near the origin. If we put $|\phi\rangle = |0\rangle$, then

$$q^{\Delta_\lambda}\langle\Psi^{(\lambda)}|0\rangle = \sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(v^i(d)|v_i(d))q^d = \sum_{d=0}^{\infty}(\dim \mathcal{H}_{\lambda,d})q^{\Delta_\lambda+d}. \tag{4.3.1}$$

This formal power series is the character of the integrable $\mathfrak{g}$-module $\mathcal{H}_\lambda$ and can be described by using theta functions. It is a multi-valued holomorphic function on D^*.

At the moment we only know that $\langle\Psi^{(\lambda)}|$ is an element of $\mathcal{H}_0[[q]]$. We shall show that it satisfies the formal gauge condition so that it can be regarded as an element of the completion of $\mathcal{V}_0(\mathfrak{F})$ at the origin.

To explain the formal gauge condition first we shall consider meromorphic functions on our elliptic curve $\mathcal{E}(q)$ having only poles at the origin [1]. Put

$$x(q,w) = -2\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} + \sum_{n=-\infty}^{+\infty}\frac{wq^n}{(1-wq^n)^2}, \tag{4.3.2}$$

$$y(q,w) = \sum_{n=-\infty}^{+\infty}\frac{(1+wq^n)wq^n}{(1-wq^n)^3}. \tag{4.3.3}$$

Then, if we put $x = \exp 2\pi\sqrt{-1}z$, we have

$$x(q,w) = -\frac{1}{12} - \frac{1}{4\pi^2}\wp(z),$$
$$y(q,w) = \frac{\sqrt{-1}}{8\pi^3}\wp'(z),$$

where $\wp(z)$ is the Weierstrass $\wp$ function. Moreover, we have

$$x(q,qw) = x(q,w),$$
$$y(q,qw) = y(q,w),$$
$$x(q,w^{-1}) = x(q,w),$$
$$y(q,w^{-1}) = -y(q,w)$$

and they satisfy the following equation:

$$y^2 - 4x^3 - x^2 + g_2(q)x + g_3(q) = 0, \tag{4.3.4}$$

where we put

$$x = x(q,w),$$
$$y = y(q,w),$$
$$g_2(q) = 20\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$
$$g_3(q) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{(7n^5+5n^3)q^n}{1-q^n}.$$

Note that $x(q,w)$ has pole of order 2 at [1] and $y(q,w)$ has pole of order 3 at [1]. They are holomorphic except at [1]. By the theory of elliptic functions we know that $H^0(\mathcal{E}(q),\mathcal{O}(*[1]))$

is spanned by monomials of $x(q,w)$ and monomials of $x(q,w)$ times $y(q,w)$. Note that $x(q,w)$ has an expansion

$$x(q,w) = \frac{w}{(1-w)^2} + \sum_{\ell=1}^{\infty}\Big\{\sum_{m|\ell}(mw^m + mw^{-m} - 2)\Big\}q^\ell.$$

This formula shows that each coefficient of q^ℓ can be regarded as a meromorphic function on $\mathbb{P}^1$ having only poles at the points ∞, 1 and 0. The same is true if we expand any polynomial $P(x,y)$ of $x(q.w)$ and $y(q,w)$ in q. That is, if we have an expansion

$$P(x(q,w),y(q,w)) = Q_1(x(q,w)) + Q_2(x(q,w))y(q,w) = \sum_{k=0}^{\infty} f_k(w)q^k, \tag{4.3.5}$$

all coefficients $f_k(w)$ are in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*(\infty + 1 + 0))$.

Now the gauge condition can be calculated formally as follows.

$$\begin{aligned}
&\langle\Psi^{(\lambda)}|X\otimes P(x(q,w),y(q,w))\phi\rangle \\
&\quad= \sum_{d=0}^{\infty}\Big\langle\Psi_d\Big|\sum_{k=0}^{\infty}(X\otimes f_k(w))q^k\Big|\phi\Big\rangle q^d \\
&\quad= \sum_{d,k=0}^{\infty}\Big(\sum_{i=1}^{m_d}\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle\Big)q^{d+k}.
\end{aligned} \tag{4.3.6}$$

By the gauge condition on $\langle\Psi| \in V_{\lambda^\dagger,0,\lambda}$, we have

$$\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle = 0.$$

This proves formally

$$\langle\Psi^{(\lambda)}|X\otimes P(x(q,w),y(q,w)) = \sum_{d,k=0}^{\infty}\Big(\sum_{i=1}^{m_d}\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle\Big)q^{d+k}. \tag{4.3.7}$$

Since we do not know the convergence of $\langle\Psi^{(\lambda)}|$, the above argument does not prove the fact

$$\langle\Psi^{(\lambda)}| \in \mathcal{V}_0^\dagger(\mathfrak{F}).$$

To go further we need to use certain results from algebraic geometry. First we take the completion $\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}$ of our sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$ at the origin. Let $\mathfrak{m}$ be the defining ideal sheaf of the point $q=0$ in D. Then, the completion is defined by

$$\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} = \varprojlim_{n\to\infty}\mathcal{V}_0^\dagger(\mathfrak{F})/\mathfrak{m}^n\mathcal{V}_0^\dagger(\mathfrak{F}).$$

Let $\widehat{\mathcal{O}}_{D/0}$ be the completion of the structure sheaf $\mathcal{O}_D$ of D at the origin. Since $\widehat{\mathcal{O}}_{D/0}$ is faithfully flat over $\mathcal{O}_D$, we have

$$\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} = \mathcal{V}_0^\dagger(\mathfrak{F})\otimes_{\mathcal{O}_D}\widehat{\mathcal{O}}_{D/0}.$$

On the other hand, we can show that $\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}$ is characterized as a subsheaf of $\mathcal{H}_0\otimes_{\mathcal{O}_D}\widehat{\mathcal{O}}_{D/0}$ by the formal gauge condition (4.3.7). Hence, we have

$$\langle\Psi^{(\lambda)}| \in \widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}.$$

It is also easy to show that $\langle \Psi^{(\lambda)}|$, $\lambda \in P_\ell$ are linearly independent over $\widehat{\mathcal{O}}_{D/0}$. This means that

$$\dim_K \widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} \otimes K \geqslant \dim \mathcal{V}_0^\dagger((\mathcal{E}(0), o(0), u)), \tag{4.3.8}$$

where K is the field of fraction of $\widehat{\mathcal{O}}_{D/0}$. Note that we have already proved that the sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$ is locally free over D^* and we can show that

$$\operatorname{rank} \mathcal{V}_0^\dagger(\mathfrak{F})|_{D^*} = \dim_K \widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} \otimes K. \tag{4.3.9}$$

Now we need to consider the sheaf $\mathcal{V}_0(\mathfrak{F})$. In Lemma 4.1.3 we established the isomorphism

$$\mathcal{V}_0(\mathfrak{F}) \otimes \mathbb{C}_0 \simeq \mathcal{V}_0((\mathcal{E}(0), o(0), u)). \tag{4.3.10}$$

A similar isomorphism exists for $\mathcal{V}_0^\dagger(\mathfrak{F}) \otimes \mathbb{C}_0$, but this can only be proved after we have proved that $\mathcal{V}_0^\dagger(\mathfrak{F})$ is locally free at the origin. This is one of the main reason we need to introduce the sheaf of covacua. ¿From (4.3.8), (4.3.9) and (4.3.10) we conclude

$$\operatorname{rank} \mathcal{V}_0(\mathfrak{F})|_{D^*} \geqslant \dim \mathcal{V}_0((\mathcal{E}(0), o(0), u)) = \dim \mathcal{V}_0^\dagger(\mathfrak{F}) \otimes \mathbb{C}_0. \tag{4.3.11}$$

But, since our sheaf $\mathcal{V}_0(\mathfrak{F})$ is coherent, we have the opposite inequality:

$$\dim \mathcal{V}_0^\dagger(\mathfrak{F}) \otimes \mathbb{C}_0 \geqslant \operatorname{rank} \mathcal{V}_0(\mathfrak{F})|_{D^*}. \tag{4.3.12}$$

Therefore equality holds in (4.3.12). This means that our sheaf $\mathcal{V}_0(\mathfrak{F})$ is a locally free $\mathcal{O}_D$-module, hence so is $\mathcal{V}_0^\dagger(\mathfrak{F})$. This proves local freeness.

In the next section we shall generalize the above arguments to prove local freeness for any sheaf of vacua $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F})$. The essential ideas of the argument in the next section are the same as above.

The above arguments are enough to prove local freeness. In Chapter 5 we shall consider the connection on $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F})$. Here we explain briefly how we can define the connection on the sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$.

Let us consider the exact sequence given in (4.2.6)

$$0 \longrightarrow \pi_* \Theta_{\mathcal{E}/D}(*o(D)) \xrightarrow{t} \mathcal{L}(\mathfrak{F}) \xrightarrow{\theta} \Theta_D(-\log 0) \longrightarrow 0.$$

The sheaf $\Theta_D(-\log 0)$ is a free $\mathcal{O}_D$-module generated by $q\frac{d}{dq}$. As we proved in Example 1.2.8, we have

$$\theta\left(u^{-1}\frac{d}{du}\right) = -q\frac{d}{dq}.$$

To introduce the connection on $\mathcal{V}_0^\dagger(\mathfrak{F})$ with a regular singularity at the origin is equivalent to giving the action of $\Theta_D(-\log 0)$ on $\mathcal{V}_0^\dagger(\mathfrak{F})$. For that purpose it is enough to show that the image of the mapping t acts trivially on $\mathcal{V}_0^\dagger(\mathfrak{F})$.

By (4.2.14) for any local section $\langle \Psi|$ of $\mathcal{V}_0^\dagger(\mathfrak{F})|_{D^*}$ and $\underline{\ell} \in \mathcal{L}(\mathfrak{F})$, we have

$$\{D(\underline{\ell})|\langle \Psi|\}\phi\rangle = -\langle \Psi|\{D(\underline{\ell})\phi\rangle\} + \theta(\underline{\ell})\Psi|\phi\rangle, \tag{4.3.13}$$

for any local section $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$. By (4.2.8) we also have

$$D(\underline{\ell})|\phi\rangle = \theta(\underline{\ell})|\phi\rangle - T[\underline{\ell}]|\phi\rangle.$$

If $\underline{\ell}$ is in the image of t we have $\theta(\underline{\ell}) = 0$. Hence, it is enough to show

$$\langle \Psi|T[\underline{\ell}]|\phi\rangle = 0.$$

That is

$$\operatorname*{Res}_{u=0}\{\ell(u)\langle\Psi|T[u]|\phi\rangle du\} = 0, \tag{4.3.14}$$

where $\underline{\ell} = \ell(u)\frac{d}{du}$.

As was shown in section 3.6, the form

$$\langle\Psi|T[z]|\phi\rangle dz^2$$

is a single valued meromorphic form. In our coordinate the form can be expressed in the form

$$\langle\Psi|T[w]|\phi\rangle(dw/w)^2,$$

where

$$w = \exp 2\pi\sqrt{-1}z.$$

Now in our case, the form $\ell(u)\langle\Psi|T[u]|\phi\rangle du$ is a global meromorphic one-form

$$\iota_\tau(\langle\Psi|T[w]|\phi\rangle(dw/w)^2),$$

which has poles only at $o(D)$, where ι is the interior product, since we have

$$\underline{\ell} = \theta(\tau)$$

for a relative vector field $\tau \in \pi_*\Theta_{\mathcal{E}/D}(*o(D))$, having pole only along $o(D)$. Hence (4.3.14) holds. Thus we can define the connection on $\mathcal{V}_0^\dagger(\mathfrak{F})$. We can show that the connection is flat. If the genus of a curve is greater than one, the form $\langle\Psi|T[z]|\phi\rangle dz^2$ is no longer a global form on the curve. (See Theorem 3.4.3, (2). Therefore, in general we can only show that the image of t acts on $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F})$ as the multiplication of a certain holomorphic functions. This is the reason why we only have a projectively flat connection in general. The details will be explained in section 5.1.

Finally we need to show that our formal solutions actually converge so that we have

$$\langle\Psi^{(\lambda)}| \in \mathcal{V}_0(\mathfrak{F}).$$

Put

$$\langle\widetilde{\Psi}^{(\lambda)}| = q^{\Delta_\lambda}\langle\Psi^{(\lambda)}|.$$

We can show that $\langle\widetilde{\Psi}^{(\lambda)}|$ satisfies the differential equation

$$q\frac{d}{dq}\langle\widetilde{\Psi}^{(\lambda)}|\Psi\rangle - \langle\widetilde{\Psi}^{(\lambda)}|L_{-2}\phi\rangle = 0, \tag{4.3.15}$$

for any $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$.

It is rather messy to prove this by direct calculation. In section 5.3 we shall give a general method to prove this fact. Here we only prove the equation (4.3.15) when $|\phi\rangle = |0\rangle$. In this case, we have

$$\begin{aligned}\langle\widetilde{\Psi}^{(\lambda)}|0\rangle &= \sum_{d=0}^{\infty}\Big(\sum_{i=1}^{m_d}\langle\Psi|v^i(d)\otimes v_i(d)\otimes 0\rangle\Big)q^{\Delta_\lambda+d}\\ &= \sum_{d=0}^{\infty}(\dim\mathcal{H}_{\lambda,d})q^{\Delta_\lambda+d}.\end{aligned}$$

is satisfied for all $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$.

First let us calculate

$$\langle\widetilde{\Psi}^{(\lambda)}|L_{-2}0\rangle.$$

For that purpose it is enough to calculate

$$\langle\Psi|v^i(d)\otimes v_i(d)\otimes L_{-2}0\rangle$$
$$=\frac{1}{2(g^*+\ell)}\sum a=1^{\dim\mathfrak{g}}\sum_{m=-\infty}^{\infty}\langle\Psi|v^i(d)\otimes v_i(d)\otimes J^a(m)J^a(-2-m)0\rangle.$$

For $m\neq -1$, since m or $-2-m$ is non-negative, we have

$$J^a(m)J^a(-2-m)|0\rangle=J^a(-2-m)J^a(m)0\rangle=0.$$

For $m=-1$ the gauge condition implies

$$\begin{aligned}\langle\Psi|v^i(d)&\otimes v_i(d)\otimes J^a(-1)J^a(-1)0\rangle\\&=\left\langle\Psi\left|v^i(d)\otimes v_i(d)\otimes J^a\left[\tfrac{1}{\xi-1}\right]J^a\left[\tfrac{1}{\xi-1}\right]0\right.\right\rangle\\&=\left\langle\Psi\left|J^a\left[\tfrac{\eta}{1-\eta}\right]J^a\left[\tfrac{\eta}{1-\eta}\right]v^i(d)\otimes v_i(d)\otimes 0\right.\right\rangle\\&\quad+2\left\langle\Psi\left|J^a\left[\tfrac{\eta}{1-\eta}\right]v^i(d)\otimes J^a\left[\tfrac{1}{\xi-1}\right]v_i(d)\otimes 0\right.\right\rangle\\&\quad+\left\langle\Psi\left|v^i(d)\otimes J^a\left[\tfrac{1}{\xi-1}\right]J^a\left[\tfrac{1}{\xi-1}\right]v_i(d)\otimes 0\right.\right\rangle.\end{aligned}$$

We have

$$\begin{aligned}\left\langle\Psi\left|J^a\left[\tfrac{\eta}{1-\eta}\right]J^a\left[\tfrac{\eta}{1-\eta}\right]v^i(d)\otimes v_i(d)\otimes 0\right.\right\rangle&=\sum_{m,n=1}^{\infty}\langle\Psi|J^a[\eta^m]J^a[\eta^n]v^i(d)\otimes v_i(d)\otimes 0\rangle\\&=\sum_{m,n=1}^{\infty}(J^a(m)J^a(n)v^i(d)|v_i(d))=0,\end{aligned}$$

since $J^a(m)J^a(n)v^i(d)\notin\mathcal{H}_{\lambda^\dagger,d}$. Similarly we have

$$\begin{aligned}\left\langle\Psi\left|v^i(d)\otimes J^a\left[\tfrac{1}{\xi-1}\right]J^a\left[\tfrac{1}{\xi-1}\right]v_i(d)\otimes 0\right.\right\rangle&=\sum_{m,n=0}^{\infty}(v^i(d)|J^a(m)J^a(n)v_i(d))\\&=(v^i(d)|J^aJ^av_i(d)).\end{aligned}$$

Also we have

$$\begin{aligned}\left\langle\Psi\left|J^a\left[\tfrac{\eta}{1-\eta}\right]v^i(d)\otimes J^a\left[\tfrac{1}{\xi-1}\right]v_i(d)\otimes 0\right.\right\rangle&=-\sum_{m,n=0}^{\infty}(J^a(m+1)v^i(d)|J^a(n)v_i(d))\\&=-\sum_{n=1}^{\infty}(J^a(n)v^i(d)|J^a(n)v_i(d))\\&=\sum_{n=1}^{\infty}(v^i(d)|J^a(-n)J^a(n)v_i(d)).\end{aligned}$$

Thus we conclude

$$\langle \Psi | v^i(d) \otimes v_i(d) \otimes L_{-2}0 \rangle = (v^i(d)|L_0 v_i(d)) = (\Delta_\lambda + d)(v^i(d)|v_i(d)).$$

Thus we obtain

$$\langle \widetilde{\Psi}^{(\lambda)} | L_{-2}0 \rangle = \sum_{d=0}^{\infty} (\Delta_\lambda + d)(\dim \mathcal{H}_{\lambda,d}) q^{\Delta_\lambda + d}.$$

The right hand side of the formula is equal to

$$q \frac{q}{dq} \langle \widetilde{\Psi}^{(\lambda)} | 0 \rangle.$$

Thus we obtain the desired result. The argument can be generalized for general $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$. In section 5.3 we shall give a more conceptual proof of (4.3.15), which can be applied to the general situation.

Remark 4.3.1. In the Physics literature, $\langle \widetilde{\Psi}^{(\lambda)} | \phi \rangle$ is written as

$$\mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0} \langle \Psi | * \otimes * \otimes \phi \rangle).$$

Actually, usually physicists add the term $q^{-\frac{c_v}{24}}$:

$$\mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0 - \frac{c_v}{24}} \langle \Psi | * \otimes * \otimes \phi \rangle). \tag{4.3.16}$$

This only changes the differential equation (4.3.15) to

$$q \frac{d}{dq} \langle \widetilde{\Psi}^{(\lambda)} | \Psi \rangle - \langle \widetilde{\Psi}^{(\lambda)} | L_{-2} \phi \rangle - \frac{c_v}{24} \langle \widetilde{\Psi}^{(\lambda)} | \Psi \rangle = 0. \tag{4.3.17}$$

The expressions (4.3.16) and (4.3.17) can be interpreted as follows. If we fix an element $|\phi\rangle \in \mathcal{H}_0$, then

$$u^\dagger(d) \otimes u(d) \otimes \longmapsto \langle \Psi | u^\dagger \otimes u \otimes \phi \rangle \tag{4.3.18}$$

may be regarded as an element of $\mathrm{Hom}(\mathcal{H}_{\lambda,d}, \mathcal{H}_{\lambda,d})$. Hence,

$$\sum_{i=1}^{\infty} \langle \Psi | v^i(d) \otimes v_i(d) \otimes \phi \rangle$$

can be interpreted as the trace on $\mathcal{H}_{\lambda,d}$ of the mapping in $\mathrm{Hom}(\mathcal{H}_{\lambda,d}, \mathcal{H}_{\lambda,d})$ associated with the mapping (4.3.18). Moreover, since L_0 acts on $\mathcal{H}_{\lambda,d}$ by the operator $\Delta_\lambda + d$, we may justify the notation

$$\mathrm{tr}_{\mathcal{H}_{\lambda,d}}(q^{L_0}) = q^{\Delta_\lambda + d}.$$

Hence, in (4.3.16) and (4.3.17) we may regard $\mathrm{tr}_{\mathcal{H}_\lambda}$ as $\sum_{d=0}^{\infty} \mathrm{tr}_{\mathcal{H}_{\lambda,d}}$. □

Remark 4.3.2. The rational mapping

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathbb{P}^2 \times D \\ [(q,(w:1)] &\longmapsto ((1 : x(q,w) : y(q,w)), q) \end{aligned}$$

gives a projective imbedding of $\mathcal{E}$ as a submanifold in $\mathbb{P}^2 \times D$ defined by the equation (4.3.4).

4.4. Local Freeness II (the General Case) and Factorization.

In this section we complete the proof of the local freeness of the sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$.

Let $\mathfrak{F}^{(0)} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N)$ be a versal family of N-pointed stable curves and $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ be a family over $\mathfrak{F}^{(0)}$ with formal neighbourhoods. Let D be the discriminant locus of the family $\mathfrak{F}^{(0)}$ and let $D = \bigcup D_i$ be the decomposition into irreducible components.

Choose D_i and put $E = D_i$. Denote by $\pi_E\colon \mathcal{C}_E \to E$ the restriction of $\mathcal{C}$ to E. Let $\widetilde{\pi}_E\colon \widetilde{\mathcal{C}}_E \to E$ be obtained by the simultaneous normalization of $\pi_E\colon \mathcal{C}_E \to E$ along the locus of double points over E and $\sigma', \sigma''\colon E \to \widetilde{\mathcal{C}}_E$ the cross-sections corresponding to the normalized double points over the locus E.

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}_E & \longrightarrow & \mathcal{C}_E & \hookleftarrow & \mathcal{C} \\ {\scriptstyle s_j,\sigma',\sigma''}\nwarrow\searrow{\scriptstyle \widetilde{\pi}_E} & & {\scriptstyle \pi_E}\downarrow\uparrow{\scriptstyle s_j,\sigma} & & \downarrow \\ & & E & \hookleftarrow & \mathcal{B} \end{array}$$

Proposition 4.4.1. *The family*

$$\widetilde{\mathfrak{F}}^{(0)}_E := (\widetilde{\pi}_E\colon \widetilde{\mathcal{C}}_E \to E; s_1, \ldots, s_N, \sigma', \sigma'')$$

is a local universal family of $(N+2)$-pointed (not necessarily connected) stable curves. If $\widetilde{\mathcal{C}}_E$ is connected, then $\widetilde{\mathfrak{F}}^{(0)}_E$ is a family of stable curves of genus $g-1$ and if it is not connected, then the family consists of two disconnected families over E of $N'+1$ pointed and $N''+1$ pointed stable curves of genus g' and g'', respectively, with $N'+N''=N$ and $g'+g''=g$.

Theorem 4.4.2. *The sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module. Hence, $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is also a locally free $\mathcal{O}_{\mathcal{B}}$-module. Moreover, for any holomorphic mapping $Y \to \mathcal{B}$, we let $\mathfrak{F}_Y$ be the pull-back of the family $\mathfrak{F}$. Then, we have a canonical isomorphism*

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_Y),$$

Proof. By Corollary 4.2.4 the theorem is true if our family $\mathfrak{F}^{(0)}$ contains no singular curves (i.e. $D = \varnothing$). Therefore, we may assume that the versal family $\mathfrak{F}^{(0)}$ contains singular curves. First we consider the case $k=1$, that is, each singular curve in the family has only one double point. Hence, $E = D_1$. Put $M = 3g-4+2N$. Let $(u_1, u_2, \ldots, u_M, q)$ be coordinates of $\mathcal{B}$ where E is defined by the equation $q=0$. Moreover, choose coordinates $(u_1, \ldots, u_M, z, w)$ of $\mathcal{C}_{\mathcal{B}}$ in a neighbourhood of the locus Σ_E of the double points of the fibers of $\mathcal{C}_E$ such that $\mathcal{C}_E$ is defined by $z=w=0$ and the holomorphic mapping π is given by

$$(u_1, \ldots, u_M, z, w) \longmapsto (u_1, \ldots, u_M, zw). \tag{4.4.1}$$

Let us fix an element $\mu \in P_\ell$. The following Lemma is proved in section 2.2 (see Lemma 2.2.12).

Lemma 4.4.3. *There exists a bilinear pairing*

$$(\cdot|\cdot)\colon \mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger} \longrightarrow \mathbb{C},$$

unique up to a constant multiple such that we have

$$(X(n)u|v) + (u|X(-n)v) = 0,$$

for any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $|u\rangle \in \mathcal{H}_\mu$, $|v\rangle \in \mathcal{H}_{\mu^\dagger}$ and $(\cdot|\cdot)$ is zero on $\mathcal{H}_\mu(d) \times \mathcal{H}_{\mu^\dagger}(d')$, if $d \neq d'$.

Now let us choose a basis $\{v_1(d), \ldots, v_{m_d}(d)\}$ of $\mathcal{H}_\mu(d)$ and with respect to the above bilinear form $(\cdot|\cdot)$ we choose the dual basis $\{v^1(d), \ldots, v^{m_d}(d)\}$ of $\mathcal{H}_{\mu^\dagger}(d)$.

The sheaf $\mathcal{V}^{\dagger}_{\mu,\mu',\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is locally free over $E-\cup_{j\neq i}D_j$ by Corollary 4.2.6. Let $\langle\Psi|$ be a section of $\mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$. Let us define an element $\langle\widetilde{\Psi}|\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(E)[[q]]$ associated with $\langle\Psi|$. For that purpose first define $\langle\Psi_d|\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(E)$ by

$$\langle\Psi_d|\Phi\rangle=\sum_{i=1}^{m_d}\langle\Psi|v_i(d)\otimes v^i(d)\otimes\Phi\rangle,\quad |\Phi\rangle\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(E). \tag{4.4.2}$$

Now define $\langle\widetilde{\Psi}|\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(E)[[q]]$ by

$$\langle\widetilde{\Psi}|\Phi\rangle=\sum_{d=0}^{\infty}\langle\Psi_d|\Phi\rangle q^d. \tag{4.4.3}$$

This construction of $\langle\tilde{\Psi}|$ (precisely speaking of $q^{\Delta_\mu}\langle\tilde{\Psi}|$) is known as the sewing procedure by physicists [32]. Now we shall show that $\langle\widetilde{\Psi}|$ satisfies the formal gauge condition. To give the precise meaning of this statement, we first prove the following lemma.

Lemma 4.4.4. *There is an $\mathcal{O}_E$-module injection*

$$\pi_*\mathcal{O}_{\mathcal{C}}(*S)|_E\hookrightarrow\widetilde{\pi}_{E*}\mathcal{O}_{\bar{\mathcal{C}}_E}(*(\sigma'+\sigma''+S))[[q]],$$
$$f\longmapsto\sum_{k=0}^{\infty}f_kq^k,$$

where

$$f_k\in\widetilde{\pi}_{E*}\mathcal{O}_{\bar{\mathcal{C}}_E}(*S+k(\sigma'+\sigma'')).$$

Proof. Choose a point $P\in\mathcal{C}_E$ which is a double point of a fiber of π_E. Then, as above we choose local coordinates $(u_1,\dots,u_{M-1},z,w)$ of $\mathcal{C}$ with center P and those $(u_1,\dots,u_{M-1},q)$ of $\mathcal{B}$ with center $\pi(P)$ such that π is given by (4.4.1). Since f is holomorphic at P, we have an expansion

$$f=f(u_1,\dots,u_{M_1},z,w)=\sum_{m\geqslant 0,n\geqslant 0}f_{m,n}(u)z^mw^n.$$

Define $g_{P'}(u,q,z)$ by

$$g_{P'}(u,q,z)=f\left(u,z,\frac{q}{z}\right)=\sum_{k=0}^{\infty}g_k(u,z)q^k,$$

where

$$g_k(u,z)=\sum_{m=0}^{\infty}f_{m,k}(u)z^{m-k}. \tag{4.4.4}$$

Define also $h_{P''}(u,q,w)$ by

$$h_{P'}(u,q,w)=f\left(u,\frac{q}{w},w\right)=\sum_{k=0}^{\infty}h_k(u,w)q^k,$$

where

$$h_k(u,w)=\sum_{n=0}^{\infty}f_{k,n}(u)w^{n-k}. \tag{4.4.5}$$

For a point $Q\in\mathcal{C}_E$ which is not a double point of a fiber, we can choose local coordinates $(u_1,\dots,u_{M-1},q,z)$ of $\mathcal{C}$ with center Q such that π is given by the projection to the first M factors. Then we have an expansion

$$f(u_1,\dots,u_{M-1},q,z)=\sum_{k=0}^{\infty}f_{Q,k}(u,z)q^k.$$

It is easy to see that the data $\{g_k(u,z),h_k(u,w),f_{Q,k}(u,z)\}$ defines a local holomorphic section of the sheaf $\widetilde{\pi}_{E*}\mathcal{O}_{\bar{\mathcal{C}}_E}(*S+k(\sigma'+\sigma''))$. This proves Lemma 4.4.4. □

Lemma 4.4.5. *For an element $f \in \pi_*\mathcal{O}_{\mathcal{C}}(*S)$ let $\sum_{k=0}^{\infty} f_k q^k$ be the expansion defined in Lemma* 4.4.4. Then we have

$$\sum_{j=1}^{N} \Big\langle \widetilde{\Psi} \Big| \sum_k \rho_j(X \otimes f_k) q^k = 0.$$

That is, $\langle \widetilde{\Psi}|$ satisfies the formal gauge condition.

Proof. By definition, for any $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}(E)$ we have

$$\begin{aligned}
&\sum_{j=1}^{N} \Big\langle \widetilde{\Psi} \Big| \sum_{k=0}^{\infty} \rho_j(X \otimes f_k) q^k \Big| \Phi \Big\rangle \\
&\qquad = \sum_{k=0}^{\infty} \sum_{d=0}^{\infty} \sum_{i=1}^{m_d} q^{k+d} \sum_{j=1}^{N} \langle \Psi | \rho_j(X \otimes f_k) | v_i(d) \otimes v^i(d) \otimes \Phi \rangle \\
&\qquad = -\sum_{k=0}^{\infty} \sum_{d=0}^{\infty} \sum_{i=1}^{m_d} q^{k+d} \langle \Psi | \rho_{\sigma'}(X \otimes g_k) \\
&\qquad\qquad + \rho_{\sigma''}(X \otimes h_k) | v_i(d) \otimes v^i(d) \otimes \Phi \rangle.
\end{aligned}$$

By (4.4.4) and (4.4.5) we have

$$\begin{aligned}
\rho_{\sigma'}(X \otimes g_k) &= \sum_{m=0}^{\infty} f_{m,k}(t) \rho_{\sigma'}(X(m-k)), \\
\rho_{\sigma''}(X \otimes h_k) &= \sum_{n=0}^{\infty} f_{k,n}(t) \rho_{\sigma''}(X(n-k)).
\end{aligned}$$

Since we have

$$(X(m-k)v_i(d)|v^j(d-m+k)) + (v_i(d)|X(k-m)v^j(d-m+k)) = 0$$

by Lemma 4.4.3, we have

$$\begin{aligned}
&\sum_{i=1}^{m_d} \rho_{\sigma'}(X(m-k)) | v_i(d) \otimes v^i(d) \otimes \Phi \rangle \\
&\qquad + \sum_{j=1}^{m_{d-m+k}} \rho_{\sigma''}(X(-(m-k))) | v_j(d-m+k) \otimes v^j(d-m+k) \otimes \Phi \rangle = 0.
\end{aligned}$$

This proves Lemma 4.4.5. □

Let $\widehat{\mathcal{O}}_{\mathcal{B}/E}$ be the completion along E, that is

$$\widehat{\mathcal{O}}_{\mathcal{B}/E} \simeq \varprojlim_n \mathcal{O}_{\mathcal{B}}/I_E^n$$

where I_E is the ideal sheaf of E. Then, there is an $\mathcal{O}_E$-algebra isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{B}/E} \simeq \mathcal{O}_E[[q]].$$

In the following we identify $\widehat{\mathcal{O}}_{\mathcal{B}/E}$ with $\mathcal{O}_E[[q]]$.

Lemma 4.4.6. *Let $\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}/E}$ be a sheaf of $\mathcal{O}_E[[q]]$-module defined by the formal gauge condition:*

$$\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}/E} = \Big\{ \langle\Phi| \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(E)[[q]] \,\Big|\, \sum_{j=1}^{N} \langle\Phi| \sum_{k} \rho_j(X \otimes f_k) q^k = 0 \quad \text{for all } f \in \pi_* \mathcal{O}_{\mathcal{C}}(*S) \Big\}.$$

Then, there is an $\mathcal{O}_E[[q]]$-module isomorphism

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} \simeq \widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}/E}. \tag{4.4.6}$$

Proof. Since $\widehat{\mathcal{O}}_{\mathcal{B}/E}$ is a faithfully flat $\mathcal{O}_{\mathcal{B}}$-module, we have an isomorphism

$$\begin{aligned} &\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} \\ &\qquad = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_{\mathcal{B}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} \\ &\qquad \simeq \underline{\mathrm{Hom}}_{\widehat{\mathcal{O}}_{\mathcal{B}/E}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E}, \widehat{\mathcal{O}}_{\mathcal{B}/E}). \end{aligned}$$

Note that we have an $\mathcal{O}_E$-module isomorphism

$$\mathcal{H}_{\vec{\lambda}}(E)[[q]] \simeq \mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E}.$$

By Lemma 4.1.3 and faithful flatness of $\widehat{\mathcal{O}}_{\mathcal{B}/E}$ we have

$$\begin{aligned} &\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} \\ &\qquad \simeq \{\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E}\} / \{(\widehat{\mathfrak{g}}(\mathfrak{F}) \mathcal{H}_{\vec{\lambda}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E}\}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\widehat{\mathfrak{g}}(\mathfrak{F}) \mathcal{H}_{\vec{\lambda}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} &= \widehat{\mathfrak{g}}(\mathfrak{F})(\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E}) \\ &\simeq \widehat{\mathfrak{g}}(\mathfrak{F}) \mathcal{H}_{\vec{\lambda},E}[[q]], \end{aligned}$$

where the action of $X \otimes f \in \widehat{\mathfrak{g}}(\mathfrak{F})$ on $\mathcal{H}_{\vec{\lambda},E}[[q]]$ is given by

$$\sum_{k=0}^{\infty} \sum_{j=1}^{N} \rho_j(X \otimes f_k) q^k,$$

where $\sum f_k q^k$ is the element corresponding to f defined in Lemma 4.4.4. Hence we have

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathcal{O}}_{\mathcal{B}/E} \simeq \mathcal{H}_{\vec{\lambda}}(E)[[q]] / \widehat{\mathfrak{g}}(\mathfrak{F}) \mathcal{H}_{\vec{\lambda}}(E)[[q]].$$

This proves Lemma 4.4.6. □

Now we are ready to prove Theorem 4.4.2. As we assumed $k = 1$, $\widetilde{\mathfrak{F}}^{(0)}_E$ is a versal family of $(N+2)$-pointed smooth curves. Choose $\mathcal{B}$ small enough so that $\bigoplus_{\mu \in P_\ell} \mathcal{V}^{\dagger}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is $\mathcal{O}_E$-free. Let

$$\{\langle\Psi_1|, \ldots, \langle\Psi_n|\}$$

be an $\mathcal{O}_E$-free basis of $\oplus_{\mu \in P_\ell} \mathcal{V}^{\dagger}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$. Let

$$\{\langle\widetilde{\Psi}_1|, \ldots, \langle\widetilde{\Psi}_n|\}$$

be the elements of $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ constructed in (4.4.3) from $\{\langle\Psi_1|,\dots,\langle\Psi_n|\}$. The correspondence $\langle\Psi_i|\mapsto\langle\widetilde{\Psi}_i|$, $i=1,\dots,n$, defines an $\mathcal{O}_E$-module homomorphism

$$\iota\colon \bigoplus_{\mu\in P_\ell}\mathcal{V}^{\dagger}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)\longrightarrow\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}/E}.$$

First we show that $\langle\widetilde{\Psi}_1|,\dots,\langle\widetilde{\Psi}_n|$ are $\mathcal{O}_E[[q]]$-linearly independent. Suppose we have a relation

$$\sum_{i=1}^{n}a_i(q)\langle\widetilde{\Psi}_i|=0,\quad a_i(q)\in\mathcal{O}_E[[q]].$$

If $a_i(0)=0$ for all i, by dividing the relation by a power of q, we may assume that one of the $a_i(q)$'s, say $a_k(q)$ satisfies the condition that $a_k(0)\neq 0$. If we put $q=0$ in the above relation, we have

$$\sum_{i=1}^{n}a_i(0)\langle\Psi_i|=0.$$

Since $\{\langle\Psi_1|,\dots,\langle\Psi_n|\}$ is $\mathcal{O}_E$-linearly independent, $a_i(0)=0$ for all i. This is a contradiction. Hence, $\langle\widetilde{\Psi}_1|,\dots,\langle\widetilde{\Psi}_n|$ generates a $\mathcal{O}_E[[q]]$-free submodule of $\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}/E}$.

Now choose points $x\in E$ and $s\in\mathcal{B}\setminus E$. Then, the above argument, Corollary 4.2.6 and (4.4.6) show that

$$\begin{aligned}\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_s&=\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_s\\&\geqslant n=\sum_{\mu\in P_\ell}\operatorname{rank}\mathcal{V}^{\dagger}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E).\end{aligned}$$

Since $\widetilde{\mathfrak{F}}_E$ is a family of smooth curves at x, $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is locally free. By Lemma 4.1.3 and Corollary 3.3.5 we have

$$\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_x=\sum_{\mu\in P_\ell}\dim_{\mathbb{C}}\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)\otimes\mathbb{C}_x.$$

Hence we have

$$\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_s\geqslant\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_x.$$

On the other hand, since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is coherent and locally free on $\mathcal{B}\setminus E$, we have the inequality

$$\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_s\leqslant\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_x.$$

Hence we have the equality

$$\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_s=\dim_{\mathbb{C}}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes\mathbb{C}_x.$$

Hence $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free.

Next consider the case where $k\geqslant 2$ so that we have the decomposition $D=\bigcup_{i=1}^{k}D_i$ of the irreducible decomposition. Let D_{sing} be the singular part of the discriminant locus D. The above argument shows that $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free over $\mathcal{B}\setminus D_{sing}$.

Lemma 4.4.7. *If $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$ is a locally free $\mathcal{O}_{D_i}$-module for all i, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module.*

Proof. Put

$$\operatorname{rank}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})|_{\mathcal{B}-D_{sing}}=m.$$

Then

$$\operatorname{rank} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) = m.$$

Hence for each point $x \in \mathcal{B}$ we have

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x) = \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x = m.$$

This implies, at each point $x \in \mathcal{B}$, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is generated by m sections as $\mathcal{O}_{\mathcal{B}}$-module in a neighbourhood of x, since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a coherent $\mathcal{O}_{\mathcal{B}}$-module. On the other hand, if $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free at a point $y \in D_{sing}$, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is generated by more than m sections as $\mathcal{O}_{\mathcal{B}}$-module in a neighbourhood of the point y, since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a coherent $\mathcal{O}_{\mathcal{B}}$-module. Therefore, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free. This proves Lemma 4.4.7 $\square$

Thus the proof is reduced to proving locally freeness of the sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$. For that purpose put $E = D_i$ and let us consider the following mapping

$$\begin{aligned} j_E \colon \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E) &\longrightarrow \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E) \\ [|\phi\rangle] &\longmapsto ([|0_{\mu,\mu^\dagger} \otimes \phi\rangle]) \end{aligned}$$

where $[|\phi\rangle]$ (respectively, $[|0_{\mu,\mu^\dagger} \otimes \phi\rangle]$) is the image of $|\phi\rangle \in \mathcal{H}_{\vec{\lambda}}(E)$ (respectively, $[|0_{\mu,\mu^\dagger} \otimes \phi\rangle] \in \mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}(E)$) in $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E)$ (respectively, $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$).

Lemma 4.4.8. *The mapping j_E is well defined and surjective. Moreover, if $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is a locally free $\mathcal{O}_E$-module for every $\mu \in P_\ell$, then j_E is an isomorphism, hence $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E)$ is locally free.*

Proof. Let us assume that we have elements $|\phi\rangle$ and $|\phi'\rangle$ which satisfy the relation

$$|\phi'\rangle = |\phi\rangle + \sum_\nu X_\nu \otimes f_\nu |\psi_\nu\rangle,$$

with

$$X_\nu \otimes f_\nu \in \widehat{\mathfrak{g}}(\mathfrak{F}_E), \qquad |\psi_\nu\rangle \in \mathcal{H}_{\vec{\lambda}}(E).$$

Let $\widetilde{f}_\nu$ be the pullback of f_ν to $\widetilde{\mathfrak{F}}_E$. Since $\widetilde{f}_\nu$ is holomorphic at $\sigma'(E)$ and $\sigma''(E)$ we have

$$\rho_{\sigma'(E)}(X_\nu \otimes \widetilde{f}_\nu)|0_{\mu,\mu^\dagger}\rangle + \rho_{\sigma''(E)}(X_\nu \otimes \widetilde{f}_\nu)|0_{\mu,\mu^\dagger}\rangle = 0.$$

Hence, we have

$$|0_{\mu,\mu^\dagger} \otimes \phi'\rangle = |0_{\mu,\mu^\dagger} \otimes \phi\rangle + \sum_\nu X_\nu \otimes \widetilde{f}_\nu |0_{\mu,\mu^\dagger} \otimes \psi_\nu\rangle.$$

Thus we have

$$[|0_{\mu,\mu^\dagger} \otimes \phi\rangle] = [|0_{\mu,\mu^\dagger} \otimes \phi'\rangle].$$

Hence the mapping j_E is well defined.

Next let us consider the exact sequence

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E) \xrightarrow{j_E} \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E) \longrightarrow \operatorname{Coker} j_E \longrightarrow 0.$$

By Corollary 3.3.5 there is an isomorphism

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E) \otimes \mathbb{C}_x \simeq \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E) \otimes \mathbb{C}_x.$$

Hence, $\operatorname{Coker} j_E \otimes \mathbb{C}_x = 0$. Therefore, by Nakayama's lemma $\operatorname{Coker} j_E = 0$. Hence, j_E is surjective. Finally consider an exact sequence

$$0 \longrightarrow \operatorname{Ker} j_E \longrightarrow \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E) \longrightarrow \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E) \longrightarrow 0.$$

Assume that $\bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is locally free. Then we have

$$\underline{Tor}^1_{\mathcal{O}_E}\Big(\bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E), \mathbb{C}_x\Big) = 0.$$

Hence we have an exact sequence

$$0 \longrightarrow \operatorname{Ker} j_E \otimes \mathbb{C}_x \longrightarrow \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_E) \otimes \mathbb{C}_x \longrightarrow \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_x) \otimes \mathbb{C}_x \longrightarrow 0.$$

Therefore, $\operatorname{Ker} j_E \otimes \mathbb{C}_x = 0$. Hence, again by Nakayama's lemma we conclude $\operatorname{Ker} j_E = 0$. Thus, j_E is isomorphic. This proves Lemma 4.4.8. □

Finally we prove Theorem 4.4.2 by induction on (g, N). For $g = 0$ and $N = 3$, $\mathcal{B}$ is a point. Hence, in this case the theorem is true. Assume that the theorem is true for $g = 0$ and $N \leqslant M$. Put $N = M + 1$. Then, for a local universal family $\mathfrak{F}^{(0)}$ of N-pointed stable curves of genus 0, $D \neq \varnothing$ and $\widetilde{\mathfrak{F}}^{(0)}_{D_i}$ is a disjoint union of local universal families of $N' + 1$ pointed and $N'' + 1$ pointed stable curves of genus 0 with $N' + N'' = N$. Since $N' + 1 \leqslant M$ and $N'' + 1 \leqslant M$, by induction hypothesis, $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E)$ is locally free. Hence, by the above argument the theorem is true. In this way we can show that the Theorem is true for $(0, N)$ with any positive integer N.

Finally assume that the Theorem is true for any (g, N) with $g \leqslant h$ and $N \leqslant M$. Then by a the similar argument as above the theorem is true for $(g, M + 1)$. Let $\mathfrak{F}^{(0)}$ be a local universal family of N pointed stable curves of genus $h + 1$ with $N \leqslant M$. If the discriminant locus D of the family is empty, that is, every curve in the family is non-singular, the theorem is true by Corollary 4.2.4. If $D \neq \varnothing$, then $\mathfrak{F}^{(0)}_{D_i}$ is either a local universal family of $N + 2$ pointed stable curves of genus h or disjoint union of a local universal families of $N' + 1$ pointed stable curves of genus g' and $N'' + 1$ pointed stable curves of genus g'' with $N' + N'' = N$ and $g' + g'' = h$. If $g' < h$ and $g'' < h$, then by the induction hypothesis Theorem is true. If $g' = 0$, then $N' \geqslant 2$, since the family is a family of $N' + 1$ pointed stable curves of genus 0. Hence $g'' = h$ and $N'' + 1 < N$. Therefore, by induction hypothesis again the theorem is true. This proves Theorem 4.4.2 completely.

The above Lemma 4.4.8 has also important consequence. To state it, we slightly generalized the situation. Put

$$E = \bigcap_{i=1}^{N} D_i.$$

Let $\mathfrak{F}^{(0)}_E$ be the pull-back of our family and let $\widetilde{\mathfrak{F}}^{(0)}_E$ be a family obtained by normalization along double points. Let σ'_i, σ''_i, be holomorphic sections attached to the double points corresponding to the locus D_i, $i = 1, \ldots, k$. We also introduce local coordinates $(u_1, \ldots, u_{M-k}, q_1, \ldots, q_k)$ of $\mathcal{B}$ such that D_i is defined by $q_i = 0$. Moreover, in a neighbourhood of the locus of the double points over D_i, we choose local coordinates $(u_1, \ldots, u_{M-k}, q_1, \ldots, q_{i-1}, z_i, w_i, q_{i+1}, \ldots, q_k)$ of $\mathcal{C}$ such that the morphism π is given by

$$\begin{aligned}(u_1, \ldots, u_{M-k}, q_1, \ldots, q_{i-1}, z_i, w_i, q_{i+1}, \ldots, q_k)& \\ \longmapsto (u_1, \ldots, u_{M-k}, q_1, \ldots, q_{i-1}, z_i \cdot w_i, q_{i+1}, \ldots, q_k)&.\end{aligned}$$

Then, in a neighbourhood of $\sigma'(E)$ (respectively, $\sigma''_i(E)$) we choose local coordinates $(u_1, \ldots, u_k, z_i)$ (respectively, $(u_1, \ldots, u_k, w_i)$) of $\widetilde{\mathfrak{F}}^{(0)}_E$ and put

$$\begin{aligned}\widetilde{\mathfrak{F}}_E := (\pi_E \colon \mathcal{C}_E \to E; s_1|_E, \ldots, s_N|_E, \sigma'_1 \ldots, \sigma'_k, \sigma''_1, \ldots, \sigma''_k;& \\ \eta_1|\mathcal{C}_E, \ldots, \eta_N|_{\mathcal{C}_E}, z_1, \ldots, z_k, w_1, \ldots, w_k)&.\end{aligned}$$

Using this notation, Lemma 4.4.8 implies the following theorem.

Theorem 4.4.9. *There is an $\mathcal{O}_E$-module isomorphism*

$$\bigoplus_{\vec{\mu}\in P_\ell^k} \mathcal{V}^\dagger_{\vec{\mu},\vec{\mu}^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_E) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_E)$$

where $\vec{\mu} = (\mu_1, \dots, \mu_k)$, $\vec{\mu}^\dagger$ is defined by $\vec{\mu}^\dagger = (\mu_1^\dagger, \dots, \mu_k^\dagger)$

5. Projectively Flat Connections and the Sheaf of Twisted Differential Operators

5.1. Projectively Flat Connections.

Let $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N; \eta_1, \dots, \eta_N)$ be a family of N-pointed curves with formal neighbourhoods such that $\mathfrak{F}^{(0)} := (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N)$ is a versal family of N-pointed curves. We shall use the notations in Chapter 4 freely, especially the one in section 4.2.

As was proved in Proposition 4.2.3 and Proposition 4.2.8 the sheaf $\mathcal{L}(\mathfrak{F})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ from the left. Moreover, for $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$, the action defines a first order differential operator, if $\theta(\vec{\ell}) \neq 0$. In this section first we shall study the action of $\vec{\ell}$ more closely when $\theta(\vec{\ell}) = 0$. Note that by (4.2.6) we have an exact sequence

$$0 \longrightarrow \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \xrightarrow{t} \mathcal{L}(\mathfrak{F}) \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log D) \longrightarrow 0. \tag{5.1.1}$$

Lemma 5.1.1. *Assume that $\mathcal{B}$ is small enough such that we can find a symmetric bidifferential ω satisfying $\operatorname{res}^2(\omega) = 1$. Then there exists a unique $\mathcal{O}_{\mathcal{B}}$-module homomorphism*

$$a\colon \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \longrightarrow \mathcal{O}_{\mathcal{B}}$$

independent of the choice of ω, such that for any $\vec{\ell} \in t(\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S))$ in (5.1.1), we have

$$D(\vec{\ell}) = a(\vec{\ell}) \cdot \mathrm{id}$$

as a linear operator acting on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ from the left and

$$D(\vec{\ell}) = -a(\vec{\ell}) \cdot \mathrm{id}$$

as an operator acting on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ from the left.

Proof. Since $\theta(\vec{\ell}) = 0$, we have that for any $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ and $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ by (4.2.8)

$$\begin{aligned}\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} &= -\sum_{j=1}^N \langle\Psi|\rho_j(T[\underline{\ell}_j])|\Phi\rangle \\ &= -\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|\Phi\rangle d\xi_j),\end{aligned}$$

where $\underline{\ell}_j = \ell_j(\xi_j)\frac{d}{d\xi_j}$ and

$$\begin{aligned}&\langle\Psi|T(\xi_j)|\Phi\rangle(d\xi_j)^2 \\ &\qquad = \lim_{\zeta_j\to\xi_j}\left\{\frac{1}{2(\ell+g^*)}\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(\zeta_j)J^a(\xi_j)|\Phi\rangle d\zeta_j\,d\xi_j\right. \\ &\qquad\qquad \left. -\frac{c_v}{2(\zeta_j-\xi_j)^2}\langle\Psi|\Phi\rangle d\zeta_j\,d\xi_j\right\}.\end{aligned}$$

Choose a symmetric bidifferential

$$\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$$

with $\mathrm{Res}^2(\omega) = 1$. The existence of such a bidifferential was proved in Proposition 1.4.4. Put

$$\begin{aligned}\langle \Psi|\widetilde{T}(z)|\Phi\rangle(dz)^2 &\\ = \lim_{w\to z}\Big\{ \frac{1}{2(\ell+g^*)} \sum_{a=1}^{\dim \mathfrak{g}} \langle\Psi|J^a(w)J^a(z)|\Phi\rangle dwdz &\\ - \frac{c_v}{2}\omega(w,z)\langle\Psi|\Phi\rangle dwdz\Big\}.&\end{aligned} \tag{5.1.2}$$

Also define $S_\omega(z)(dz)^2$ by

$$S_\omega(z)(dz)^2 = 6\lim_{w\to z}\left\{\omega(w,z)dwdz - \frac{dwdz}{(w-z)^2}\right\}.$$

The form $S_\omega(z)(dz)^2$ is called a projective connection. It depend not only on the choice of ω but also on the choice of local coordinates:

$$S_\omega(w)(dw)^2 = S_\omega(z)(dz)^2 - \{w;z\}(dz)^2.$$

Now we have

$$\langle\Psi|T(\xi_j)|\Phi\rangle(d\xi_j)^2 = \langle\Psi|\widetilde{T}(\xi_j)|\Phi\rangle(d\xi_j)^2 + \frac{c_v}{12}\langle\Psi|\Phi\rangle S_\omega(\xi_j)(d\xi_j)^2.$$

Thus we have

$$\begin{aligned}\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = -\sum_{j=1}^{N}\mathrm{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|\widetilde{T}(\xi_j)|\Phi\rangle\Phi d\xi_j)&\\ - \frac{c_v}{12}\langle\Psi|\Phi\rangle\sum_{j=1}^{N}\mathrm{Res}_{\xi_j=0}(\ell_j(\xi_j)S_\omega(\xi_j)d\xi_j).&\end{aligned} \tag{5.1.3}$$

Since $\ell_j(z)\langle\Psi|\widetilde{T}(z)|\Phi\rangle dz$ is a global meromorphic one form which has poles only at $s_j(\mathcal{B})$, the first term of the right hand side of the above equality is zero. Therefore, if we put

$$a_\omega(\vec{\ell}) = -\frac{c_v}{12}\sum_{j=1}^{N}\mathrm{Res}_{\xi_j=0}(\ell_j(\xi_j)S_\omega(\xi_j)d\xi_j). \tag{5.1.4}$$

Then we have

$$\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = a_\omega(\vec{\ell})\langle\Psi|\Phi\rangle.$$

Now since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ are locally free and dual to each other, we conclude that

$$D(\vec{\ell})|\Phi\rangle = a_\omega(\vec{\ell})|\Phi\rangle.$$

Let us show that $a_\omega(\vec{\ell})$ is independent of the choice of ω. If ω' is another symmetric bidifferential in $H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ with $\mathrm{Res}^2(\omega) = 1$, then $\omega - \omega'$ is a holomorphic bidifferential on $\mathcal{C}\times_{\mathcal{B}}\mathcal{C}$. Hence, $S_\omega - S_{\omega'}$ is also a holomorphic section of $\omega^{\otimes 2}_{\mathcal{C}/\mathcal{B}}$ on $\mathcal{C}$. Let τ be an element of $\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S)$ with $t(\tau) = \vec{\ell}$. Then, $\tau(z)(S_\omega(z) - S_{\omega'}(z))dz$ is a meromorphic one-form on $\mathcal{C}$. Hence, we have

$$\sum_{j=1}^{N}\mathrm{Res}_{\xi_j=0}(\ell(\xi_j)(S_\omega(\xi_j) - S_{\omega'}(\xi_j))d\xi_j) = 0.$$

Thus, we conclude

$$a_\omega(\vec{\ell}) = a_{\omega'}(\vec{\ell}).$$

This shows the first part of the lemma. Similarly we can prove the second part by (4.2.14) . □

From the above proof we also get the following corollary.

Corollary 5.1.2. *Under the same assumption as in Lemma* 5.1.1 *we have that if we for each* $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ *define* $a_\omega(\vec{\ell})$ *by* (5.1.4), *then*

$$a_\omega : \mathcal{L}(\mathfrak{F}) \longrightarrow \mathcal{O}_{\mathcal{B}}$$

is an $\mathcal{O}_{\mathcal{B}}$*-module homomorphism.*

Remark 5.1.3. In the above corollary, since $\vec{\ell}$ is not necessarily an image of a global meromorphic vector field, $a_\omega(\vec{\ell})$ does depend on the choice of ω.

Now we are ready to define the connections on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$, if $\mathcal{B}$ is small enough. Let us fix a symmetric bidifferential $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ with $\mathrm{Res}^2(\omega) = 1$. For each element $X \in \Theta_{\mathcal{B}}(-\log D)$, there is an element $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}) = X$. Define an operator $\nabla^{(\omega)}_X$ acting on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ from the left by

$$\nabla^{(\omega)}_X([|\Phi\rangle]) = D(\vec{\ell})([|\Phi\rangle]) - a_\omega(\vec{\ell})([|\Phi\rangle]), \tag{5.1.5}$$

where $[|\Phi\rangle]$ denotes the element in $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ corresponding to $|\Phi\rangle$.

Proposition 5.1.4. $\nabla^{(\omega)}_X$ *is well defined and enjoys the following property*

$$\nabla^{(\omega)}_X([f|\Phi\rangle]) = X(f)[|\Phi\rangle] + f\nabla^{(\omega)}_X([|\Phi\rangle]), \tag{5.1.6}$$

for all $f \in \mathcal{O}_{\mathcal{B}}$. *Hence, the correspondence* $X \mapsto \nabla^{(\omega)}_X$ *defines a connection on* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ *with regular singularities along* D.

Proof. Choose another $\vec{\ell}' \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}') = X$. Then, we have

$$a(\vec{\ell} - \vec{\ell}') = a_\omega(\vec{\ell}) - a_\omega(\vec{\ell}').$$

By Lemma 5.1.1 we have

$$(D(\vec{\ell}) - D(\vec{\ell}'))[|\Phi\rangle] = -\sum_{j=1}^{N} a_\omega(\vec{\ell} - \vec{\ell}')[|\Phi\rangle].$$

Thus, as operators on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ we have the equality

$$D(\vec{\ell}) - a_\omega(\vec{\ell}) \cdot \mathrm{id} = D(\vec{\ell}') - a_\omega(\vec{\ell}') \cdot \mathrm{id}.$$

Hence, $\nabla^{(\omega)}_X$ does not depend on the choice of $\vec{\ell}$. By Proposition 4.2.2, (3) the equality (5.1.6) holds. □

Theorem 5.1.5. *The connection* $\nabla^{(\omega)}$ *on* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})|_{\mathcal{B}-D}$ *is projectively flat.*

Proof. Put $\mathcal{B}' = \mathcal{B} \setminus D$. For elements $X, Y \in \Theta_{\mathcal{B}'}$ choose $\vec{\ell}, \vec{m} \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}) = X$ and $\theta(\vec{m}) = Y$. For simplicity of notation, in the proof we drop the superscript (ω) in the notation of the connection. The curvature is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

By a direct calculation we have

$$\begin{aligned}
&\nabla_X(\nabla_Y([|\Phi\rangle]) \\
&\quad = D(\vec{\ell})(D(\vec{m})[|\Phi\rangle]) - a_\omega(\vec{m})[|\Phi\rangle]) - a_\omega(\vec{\ell})(D(\vec{m})[|\Phi\rangle] - a_\omega(\vec{m})[|\Phi\rangle]) \\
&\quad = D(\vec{\ell})D(\vec{m})[|\Phi\rangle] - X(a_\omega(\vec{m}))[|\Phi\rangle] - a_\omega(\vec{m})(D(\vec{\ell})[|\Phi\rangle]) \\
&\qquad\qquad - a_\omega(\vec{\ell})(D(\vec{m})[|\Phi\rangle]) + a_\omega(\vec{\ell})a_\omega(\vec{m})[|\Phi\rangle].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}&[\nabla_X,\nabla_Y][|\Phi\rangle]\\&\quad=[D(\vec{\ell}),D(\vec{m})][|\Phi\rangle]-X(a_\omega(\vec{m}))\cdot[|\Phi\rangle]+Y(a_\omega(\vec{\ell}))\cdot[|\Phi\rangle].\end{aligned}$$

On the other hand, by (4.2.7) for $\theta(\vec{n})=[X,Y]$ we have

$$\vec{n}=[\vec{\ell},\vec{m}]_d=[\vec{\ell},\vec{m}]_0+X(\vec{m})-Y(\vec{\ell})$$

and

$$D(\vec{n})=[D(\vec{\ell}),D(\vec{m})]+\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\left(\frac{d^3\ell_j}{d\xi_j^3}m_j\,d\xi_j\right)\cdot\mathrm{id}\,.$$

Hence, we have

$$\begin{aligned}\nabla_{[X,Y]}[|\Phi\rangle]=[D(\vec{\ell}),D(\vec{m})][|\Phi\rangle]-a_\omega(\vec{n})\cdot[|\Phi\rangle]\\-\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\left(\frac{d^3\ell_j}{d\xi_j^3}m_j\,d\xi_j\right)\cdot[|\Phi\rangle].\end{aligned}$$

Put

$$A(\vec{\ell},\vec{m})=\Big\{a_\omega(\vec{n})-X(a_\omega(\vec{m}))+Y(a_\omega(\vec{\ell}))+\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\Big(\frac{d^3\ell_j}{d\xi_j^3}m_j\,d\xi_j\Big)\Big\}\,\mathrm{id}\,.$$

By definition we have

$$a_\omega(\vec{m})=-\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}(m_j(\xi_j)S_\omega(\xi_j)d\xi_j).$$

Hence, we have

$$\begin{aligned}A(\vec{\ell},\vec{m})=\frac{c_v}{12}\Big\{&\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\left(m_j(\xi_j)\ell_j'(\xi_j)S_\omega(\xi_j)+m_j(\xi_j)X\left(S_\omega(\xi_j)\right)d\xi_j\right)\\&-\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\left(\left(\ell_j(\xi_j)m_j'(\xi_j)S_\omega(\xi_j)+\ell_j(\xi_j)Y\left(S_\omega(\xi_j)\right)d\xi_j\right)\right)\\&+\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\Big(\frac{d^3\ell_j}{d\xi_j^3}m_j\,d\xi_j\Big)\Big\}\,\mathrm{id},\end{aligned}$$

where $\ell_j'(\xi_j)$ and $m_j'(\xi_j)$ are the derivation with respect to the variable ξ_j. Since $A(\vec{\ell},\vec{m})$ is equal to $R(X,Y)$, $A(\vec{\ell},\vec{m})$ is independent of the choice of liftings of $\vec{\ell}$ and $\vec{m}$ to $\mathcal{L}(\mathfrak{F})$. Thus, the curvature form R may be regarded as a holomorphic two form, hence the connection is projectively flat. □

Note that our connection does depend on the choice of the bidifferential ω. In the next section we shall study the difference more closely. Here, we only calculate the difference as operators. Let $\omega'\in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C},\omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ be another symmetric bidifferential with $\mathrm{Res}^2(\omega)$. By (5.1.5) and (5.1.6) we have

$$\begin{aligned}\nabla_X^{(\omega)}-\nabla_X^{(\omega')}&=(a_\omega(\vec{\ell})-a_\omega(\vec{\ell}))\cdot\mathrm{id}\\&=-\frac{c_v}{12}\sum_{j=1}^N\operatorname*{Res}_{\xi_j=0}\left(\ell_j(\xi_j)\left(S_\omega(\xi_j)-S_{\omega'}(\xi_j)\right)d\xi_j\right)\cdot\mathrm{id},\end{aligned}\tag{5.1.7}$$

where $\theta(\vec{\ell}) = X$. Since $S_\omega(z)dz^2 - S_{\omega'}(z)dz^2$ is holomorphic, the last term of (5.1.7) is independent of the choice of the lifting $\vec{\ell}$ of the vector field X and defines a holomorphic one form $\phi_{\omega,\omega'}$ with

$$\nabla_X^{(\omega)} - \nabla_X^{(\omega')} = -\frac{c_v}{12}\langle \phi_{\omega,\omega'} X\rangle, \tag{5.1.8}$$

where $\langle\cdot\rangle$ is the canonical pairing of holomorphic one forms and holomorphic vector fields.

In the lecture notes the dual connection of $\nabla^{(\omega)}$ on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ is considered as a connection acting on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ from the left and we use the same notation $\nabla^{(\omega)}$. Thus the connection on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ is given by

$$\nabla_X^{(\omega)}\langle\Psi| = D(\vec{\ell})(\langle\Psi|) + a_\omega(\vec{\ell})\langle\Psi|, \tag{5.1.9}$$

where $\theta(\vec{\ell}) = X$. This defines an integrable connection on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$. A different choice of ω' gives the following difference.

$$\nabla_X^{(\omega)} - \nabla_X^{(\omega')} = \frac{c_v}{12}\langle \phi_{\omega,\omega'} X\rangle. \tag{5.1.10}$$

Note that the difference in signs between (5.1.8) and (5.1.10).

Remark 5.1.6. In [38] we define the connection by introducing the differential operator $D(\ell)$ on the universal family

$$\widetilde{\mathfrak{F}} = (\widetilde{\pi}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}; \widetilde{s}_1, \dots, \widetilde{s}_N; \widetilde{\eta}_1, \dots, \widetilde{\eta}_N)$$

of N-pointed stable curves with formal coordinates and then we reduce it to the corresponding versal family

$$\widetilde{\mathfrak{F}}^{(1)} = (\widetilde{\pi}^{(1)}\colon \widetilde{\mathcal{C}}^{(1)} \to \widetilde{\mathcal{B}}^{(1)}; \widetilde{s}_1^{(1)}, \dots, \widetilde{s}_N^{(1)}; \widetilde{\eta}_1^{(1)}, \dots, \widetilde{\eta}_N^{(1)})$$

of N-pointed stable curves with first order neighbourhoods. The arguments in [38] can be used to define the projectively flat connection on the bundle of conformal blocks on the base space $\widetilde{\mathcal{B}}$ of the universal family of N-pointed stable curves with formal coordinates. For any family

$$\mathfrak{F}^{(1)} = (\pi^{(1)}\colon \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}; s_1^{(1)}, \dots, s_N^{(1)}; \eta_1^{(1)}, \dots, \eta_N^{(1)})$$

of N-pointed stable curves with first order neighbourhoods, there is a holomorphic mapping $\mu\colon \mathcal{B}^{(1)} \to \widetilde{\mathcal{B}}$ such that the family $\mathfrak{F}^{(1)}$ is the corresponding family of the pull-back $\mu^*\widetilde{\mathfrak{F}}$. Hence, we can induce the connection on the bundles of conformal blocks on $\mathcal{B}^{(1)}$. Moreover, for any family $\mathfrak{F}$ of N-pointed stable curve with formal neighbourhoods, we have a unique holomorphic mapping $\nu\colon B \to \widetilde{\mathcal{B}}$ of the base spaces such that the family $\mathfrak{F}$ is the pull-back $\nu^*\widetilde{\mathfrak{F}}$. The pull-back connection on the bundle of conformal blocks on B is the same as the connection defined above, if the family $\mathfrak{F}$ is versal as a family of N-pointed stable curves. Note that by Theorem 3.2.4 it is easily shown that the induced connection does only depend on the first order neighbourhoods of the family.

5.2. The Sheaf of Twisted Differential Operators.

In the preceding section we introduced integrable connections with regular singularities along the discriminant locus locally on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$. These connection was defined only locally, because we need to choose a bidifferential ω satisfying (5.1.2) to define the connections. Moreover, these connections depend on the choice of bidifferentials. In this section we shall study the dependence more closely and define the sheaf of twisted differential operators operating on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$, whose first order part is locally equal to the connection $\nabla^{(\omega)}$.

First we briefly recall the definition and elementary properties of the sheaf of twisted differential operators. Let M be a complex manifold and $D = \bigcup_{i=1}^{k} D_i$ be a divisor with normal crossings on M.

Definition 5.2.1. A sheaf $\mathcal{D}_\xi(-\log D)$ is called a *sheaf of twisted differential operators with regular singularities* along D, if there exists an exact sequence of $\mathcal{O}_\mathcal{B}$-modules and Lie algebras

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{D}_\xi(-\log D) \longrightarrow \Theta_M(-\log D) \longrightarrow 0,$$

where $\mathcal{O}_M$ is regarded as a sheaf of Abelian Lie algebras induced by the usual multiplication.

Hence the sheaf $\mathcal{D}_\xi(-\log D)$ is locally isomorphic to the sheaf $\mathcal{D}(-\log D)$ of first order holomorphic differential operators with regular singularities along D as a sheaf of Lie algebras. Therefore, if we choose a suitably fine open covering $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ of M, we can find a trivialization

$$\phi_\alpha : \mathcal{D}_\xi(-\log D)|_{\mathcal{U}_\alpha} \simeq \Theta_{\mathcal{U}_\alpha}(-\log D) \oplus \mathcal{O}_{\mathcal{U}_\alpha},$$

where ϕ_α is both an $\mathcal{O}_M$-module homomorphism and a Lie algebra homomorphism. On $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$, $\phi_{\alpha\beta} \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{U}_{\alpha\beta}}}(\Theta_{\mathcal{U}_{\alpha\beta}}, \mathcal{O}_{\mathcal{U}_{\alpha\beta}})$ is defined by

$$\phi_\alpha \circ \phi_\beta^{-1}((X, f)) = (X, f + \phi_{\alpha\beta}(X)), \tag{5.2.1}$$

where

$$(X, f) \in \Theta_{\mathcal{U}_{\alpha\beta}}(-\log D) \oplus \mathcal{O}_{\mathcal{U}_{\alpha\beta}}.$$

We may regard $\phi_{\alpha\beta}$ as a meromorphic one form on M with logarithmic singularities along $D \cap \mathcal{U}_{\alpha\beta}$ by the following isomorphism

$$H^0\left(\mathcal{U}_{\alpha\beta}, \underline{Hom}_{\mathcal{O}_M}(\Theta_M(-\log D), \mathcal{O}_M)\right) \simeq H^0(\mathcal{U}_{\alpha\beta}, \Omega^1_M(\log D)).$$

In the following we often use this identification. Let us prove that $\phi_{\alpha\beta}$ is a d-closed one form. For $X, Y \in \Theta_{\mathcal{U}_{\alpha\beta}}(-\log D)$ and $f, g \in \mathcal{O}_{\mathcal{U}_{\alpha\beta}}$, since the trivializations are Lie algebra homomorphisms, we have on $\mathcal{U}_\beta$

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f))$$

and on $\mathcal{U}_\alpha$ we have

$$\begin{aligned}\phi_{\alpha\beta}([(X, f), (Y, g)]) &= [(X, f + \phi_{\alpha\beta}(X)), (Y, g + \phi_{\alpha\beta}(Y))] \\ &= ([X, Y], (X(g) - Y(f) + X(\phi_{\alpha\beta}(Y)) - Y(\phi_{\alpha\beta}(X)))).\end{aligned}$$

Hence, we have

$$\phi_{\alpha\beta}([X, Y]) = X(\phi_{\alpha\beta}(Y)) - Y(\phi_{\alpha\beta}(X)).$$

Thus, we conclude

$$d\phi_{\alpha\beta}(X, Y) = X(\phi_{\alpha\beta}(Y)) - Y(\phi_{\alpha\beta}(X)) - \phi_{\alpha\beta}([X, Y]) = 0.$$

Let $\Omega^{1,cl}_M(\log D)$ be the sheaf of d-closed meromorphic one forms with logarithmic singularities along D, then $\{\phi_{\alpha\beta}\}$ is a cocycle and defines a cohomology class ξ in the first cohomology group $H^1(M, \Omega^{1,cl}_M(\log D))$.

Conversely if we have a cocycle $\{\phi_{\alpha\beta}\}$ defining a cohomology class in

$$H^1(M, \Omega^{1,cl}_M(\log D)),$$

then by identifying $\Theta_{\mathcal{U}_\alpha} \oplus \mathcal{O}_{\mathcal{U}_\alpha}$ and $\Theta_{\mathcal{U}_\beta} \oplus \mathcal{O}_{\mathcal{U}_\beta}$ by (5.2.1) we can define a sheaf of twisted differential operators.

Now the following proposition is an easy consequence of the above consideration.

Proposition 5.2.2. *There exists a one-to-one correspondence between the set of isomorphism classes of the sheaf $\mathcal{D}_\xi(-\log D)$ of twisted differential operators with regular singularities along D and the cohomology group*

$$H^1\left(M, \Omega_M^{1,cl}(\log D)\right).$$

Example 5.2.3. Let us assume that $D = \varnothing$. We let $\mathcal{L}$ be a line bundle over M. There is an open covering $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ of M such that we have trivializations

$$f_\alpha : \mathcal{L}|_{\mathcal{U}_\alpha} \simeq \mathcal{O}_{\mathcal{U}_\alpha}.$$

On $\mathcal{U}_{\alpha\beta}$ put

$$f_{\alpha\beta} := f_\alpha \circ f_\beta^{-1}.$$

Then the $f_{\alpha\beta}$'s are transition functions of $\mathcal{L}$. The sheaf $\mathcal{D}_\mathcal{L}$ of twisted differential operators associated with the line bundle $\mathcal{L}$ is a sheaf of first order differential operators on $\mathcal{L}$ defined by

$$\mathcal{D}_\mathcal{L}|_{\mathcal{U}_\alpha} = f_\alpha^{-1} \circ \mathcal{D}_{\mathcal{U}_\alpha} \circ f_\alpha, \tag{5.2.1}$$

where $\mathcal{D}_{\mathcal{U}_\alpha}$ is the sheaf of holomorphic first order differential operators on $\mathcal{U}_\alpha$. By (5.2.1) a Lie algebra isomorphism of $\mathcal{D}_\mathcal{L}|_{\mathcal{U}_\alpha}$ to $\mathcal{D}_{\mathcal{U}_\alpha}$ is given by f_α. Therefore, it is easy to show that the cohomology class

$$d\log f_{\alpha\beta} \in H^1(M, \Omega_M^{1,cl})$$

is the one associated with $\mathcal{D}_\mathcal{L}$.

Next we define the characteristic class of a sheaf $\mathcal{D}_\xi(-\log D)$ of twisted differential operators with regular singularities along D over M. First consider the case $D = \varnothing$. There exists an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_M \xrightarrow{d} d\mathcal{O}_M \longrightarrow 0.$$

Note that $d\mathcal{O}_M = \Omega_M^{1,cl}$. Hence, from the exact sequence we obtain a long exact sequence:

$$\longrightarrow H^1(M, \mathcal{O}_M) \longrightarrow H^1(M, \Omega_M^{1,cl}) \xrightarrow{c} H^2(M, \mathbb{C}) \longrightarrow H^2(M, \mathcal{O}_M) \longrightarrow . \tag{5.2.2}$$

The mapping c in the above exact sequence defines a characteristic class. Namely, if $\xi \in H^1(M, \Omega_M^{1,cl})$ is the cohomology class corresponding to $\mathcal{D}_\xi$, then $c(\xi) \in H^2(M, \mathbb{C})$ is the corresponding characteristic class. There is an exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}_M^* \xrightarrow{d\log} \Omega_M^{1,cl} \longrightarrow 0.$$

¿From this exact sequence we obtain a long exact sequence

$$\longrightarrow H^1(M, \mathcal{O}_M^*) \longrightarrow H^1(M, \Omega_M^{1,cl}) \xrightarrow{\overline{c}} H^2(M, \mathbb{C}^*) \longrightarrow . \tag{5.2.3}$$

On the other hand, the exponential mapping defines a natural mapping

$$\exp : H^2(M, \mathbb{C}) \longrightarrow H^2(M, \mathbb{C}^*)$$

and we have the equality

$$\overline{c} = \exp \circ c. \tag{5.2.4}$$

The following lemma is an easy consequence of the above exact sequences (5.2.2), (5.2.3) and the equality (5.2.4).

Lemma 5.2.4. *Suppose $D = \varnothing$. The sheaf $\mathcal{D}_\xi$ of twisted differential operators on M is defined by means of a line bundle $\mathcal{L}$ on M by Example* 5.2.3 *if and only if*

$$\overline{c}(\xi) = 1,$$

where $\xi \in H^1(M, \Omega_M^{1,cl})$ is the cohomology class associated with the sheaf $\mathcal{D}_\xi$.

In case $D \neq \varnothing$ we can define the characteristic map:

$$c\colon H^1\left(M, \Omega_M^{1,cl}(\log D)\right) \longrightarrow H^2(M \setminus D, \mathbb{C})$$

in a similar manner. Since we shall not use this fact later, we leave the precise definition to the reader.

Now let us come back to our original situation. Let $\mathfrak{F}^{(0)} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N)$ be a versal family of N-pointed stable curves of genus g. As was discussed in section 1.3, the family is obtained from a versal family $\mathfrak{F}^{(\varnothing)} = (\pi^{(\varnothing)}\colon \mathcal{C}^{(\varnothing)} \to \mathcal{B}^{(\varnothing)})$ by putting

$$\mathcal{B} = \overbrace{\mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \times_{\mathcal{B}^{(\varnothing)}} \cdots \times_{\mathcal{B}^{(\varnothing)}} \mathcal{C}^{(\varnothing)}}^{N} \setminus \bigcup_{i<j} \Delta_{ij},$$

$$\mathcal{C} = \mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \mathcal{B},$$

where Δ_{ij} is the (i,j)-th diagonal and π is the projection to the second factor. There are natural mappings

$$\widetilde{p}\colon \mathcal{C} \longrightarrow \mathcal{C}^{(\varnothing)},$$
$$p\colon \mathcal{B} \longrightarrow \mathcal{B}^{(\varnothing)}.$$

Note that we can always choose a bidifferential which is the pull-back of the one on an open set of $\mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \mathcal{C}^{(\varnothing)}$. Therefore, let $\{U_\alpha\}_\alpha$ be an open covering of $\mathcal{B}^{(\varnothing)}$ such that we can find symmetric bidifferentials

$$\omega_\alpha^{(\varnothing)} \in H^0\left(\pi^{(\varnothing)-1}(U_\alpha) \times_{U_\alpha} \pi^{(\varnothing)-1}(U_\alpha), \omega_{\mathcal{C}^{(\varnothing)} \times_{\mathcal{B}^{(\varnothing)}} \mathcal{C}^{(\varnothing)}/\mathcal{B}^{(\varnothing)}}(2\Delta)\right)$$

with $\mathrm{Res}^2\, \omega_\alpha^{(\varnothing)} \equiv 1$. Put

$$\mathcal{U}_\alpha = p^{-1}(U_\alpha),$$
$$\omega_\alpha = \widetilde{p}^*(\omega_\alpha^{(\varnothing)}).$$

On $\mathcal{U}_\alpha$ let us define the action of $\Theta_{\mathcal{U}_\alpha}(-\log D) \oplus \mathcal{O}_{\mathcal{U}_\alpha}$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ by

$$(X, f) \cdot [|\Phi\rangle] = \nabla_X^{(\omega_\alpha)}[|\Phi\rangle] + f \cdot [|\Phi\rangle], \tag{5.2.5}$$

where $X \in \Theta_{\mathcal{U}_\alpha}(-\log D)$, $f \in \mathcal{O}_{\mathcal{U}_\alpha}$ and $[|\Phi\rangle] \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is the element corresponding to an element $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$. (For the definition of $\nabla_X^{(\omega_\alpha)}$ see (5.1.6).) On $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \bigcap \mathcal{U}_\beta \neq \varnothing$, the difference of the actions is given by a homomorphism

$$\varphi_{\alpha\beta}\colon \Theta_{\mathcal{U}_\alpha}(-\log D)|_{\mathcal{U}_{\alpha\beta}} \longrightarrow \mathcal{O}_{\mathcal{U}_{\alpha\beta}}$$

defined by

$$\varphi_{\alpha\beta}(X) = -\frac{c_v}{12} \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0} \left(\ell_j(S_{\omega_\alpha}(\xi_j) - S_{\omega_\beta}(\xi_j))d\xi_j\right), \tag{5.2.6}$$

where $\theta(\vec{\ell}) = X \in \Theta_{\mathcal{U}_{\alpha\beta}}(-\log D)$. Now let us choose symmetric bidifferential $\omega_\alpha^{(\varnothing)}$ constructed by the prime form on the smooth fibers. (See section 1.4.) Then, on $\mathcal{U}_{\alpha\beta} \setminus D$ we have by (1.4.22) that

$$\omega_\beta^{(\varnothing)}(x,y)dxdy = \omega_\alpha^{(\varnothing)}(x,y)dxdy - \frac{1}{2}\sum_{i\leqslant j}(v_i(x)v_j(y)+v_i(y)v_j(x))\Pi^*\frac{\partial}{\partial\tau_{ij}}\log\det(C\tau+D)dxdy,$$

where Π is the period mapping form U_α to $\mathfrak{S}_g$. (See (1.4.21).) Hence, as cotangent vectors we have

$$S_{\omega_\beta}(z)dz^2 = S_{\omega_\alpha}(z)dz^2 - 6\Pi^* d\log\det(C\tau+D).$$

Thus, on $U_{\alpha\beta} \setminus D$ we have

$$\varphi_{\alpha\beta} = -\frac{c_v}{2}\Pi^* d\log\det(C\tau+D).$$

Therefore, on $U_{\alpha\beta} \setminus D$, $\phi_{\alpha\beta}$ is d-closed, hence it is d-closed on $U_{\alpha\beta}$. Thus, $\{\varphi_{\alpha\beta}\}$ defines a cohomology class in $H^1(\mathcal{B}, \Omega_{\mathcal{B}}^{1,cl}(\log D))$. Note that if we choose other bidifferentials $\{\omega'_\alpha\}$ satisfying (5.1.2), we get the same cohomology class $\{\varphi'_{\alpha\beta}\} = \{\varphi_{\alpha\beta}\}$. It is also easy to show that the cohomology class is independent of the choice of open coverings of $\mathcal{B}$. We denote this cohomology class by $\xi(\mathfrak{F})$.

Definition 5.2.6. *The sheaf $\mathcal{D}_{\mathcal{B}}(c_v)$ of twisted differential operators* is the sheaf associated with the cohomology class $\xi(\mathfrak{F})$ by Proposition 5.2.2.

Let us describe the cohomology class by using the cohomology of the mapping class group Mod_g. Let $\widetilde{\pi}\colon \mathcal{C} \to \mathbb{T}_g$ be the universal family of smooth curves of genus g over the Teichmüller space $\mathbb{T}_g$. The modular group Mod_g operates on the family $\widetilde{\pi}\colon \mathcal{C} \to \mathbb{T}_g$. Unfortunately, the quotient space $M_g := \mathbb{T}_g/\mathrm{Mod}_g$ is singular and there exists no universal family over M_g for $g \geqslant 1$. But there exists a subgroup Γ_0 of the modular group of finite index such that the versal family $\mathfrak{F}^{(\varnothing)} := (\pi\colon \mathcal{C}^{(\varnothing)} = \widetilde{\mathcal{C}}/\Gamma_0 \to \mathcal{B}_0 = \mathbb{T}_g/\Gamma_0)$ exists. Note that we have

$$H^i(\mathbb{T}_g/\Gamma_0, \mathbb{C}) = H^i(\Gamma_0, \mathbb{C}),$$
$$H^i(\mathbb{T}_g/\Gamma_0, \Omega^{1,cl}) = H^i(\Gamma_0, H^0(\mathbb{T}_g, \Omega^{1,cl}_{\mathbb{T}_g})),$$

where in the right hand side we consider $\mathbb{C}$ and $H^0(\mathbb{T}_g, \Omega^{1,cl}_{\mathbb{T}_g})$ as Γ_0-modules. Moreover, $M_g = \mathbb{T}_g/\mathrm{Mod}_g$ has the structure of a V-manifold, that is, an analytic space with only finite group quotient singularities, and we have

$$H^i(M_g, \mathbb{C}) = H^i(\mathrm{Mod}_g, \mathbb{C}),$$
$$H^i(M_g, \Omega^{1,cl}_{M_g}) = H^i(\mathrm{Mod}_g, H^0(\mathbb{T}_g, \Omega^{1,cl}_{\mathbb{T}_g})).$$

Therefore, we may consider the cohomology class $\xi(\mathfrak{F})$ of the sheaf of twisted differential operators attached to the virtual universal family $\mathfrak{F} = (\pi\colon \mathcal{C} \to M_g)$, or to the universal family over the moduli stack M_g as an element in

$$H^1\left(\mathrm{Mod}_g, H^0(\mathbb{T}_g, \Omega^{1,cl}_{\mathbb{T}_g})\right).$$

Let us calculate the class $\xi(\mathfrak{F})$. For the universal family $\widetilde{\pi}\colon \widetilde{\mathcal{C}} \to \mathbb{T}_g$, by using the prime form, we can define a symmetric bidifferential $\omega(x,y)dxdy$ as a section of the sheaf $p_1^*\omega_{\bar{\mathcal{C}}/\mathbb{T}_g} \otimes p_2^*\omega_{\bar{\mathcal{C}}/\mathbb{T}_g}(2\Delta)$ such that the restriction to each fiber gives a symmetric bidifferential with $\mathrm{Res}^2 \equiv 1$. (See (1.4.30).) The choice of the symmetric bidifferential depends on the choice of a family of symplectic basis of the universal family. More precisely, if a bidifferential ω is defined by using

a family of symplectic bases $\{\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_g,\widetilde{\beta}_1,\dots,\widetilde{\beta}_g\}$ and a bidifferential $\widehat{\omega}$ is defined by using a family of symplectic bases $\{\widehat{\alpha}_1,\dots,\widehat{\alpha}_g,\widehat{\beta}_1,\dots,\widehat{\beta}_g\}$, then we have the relation

$$\widehat{\omega}(x,y)dxdy = \omega(x,y)dxdy - \frac{1}{2}\sum_{i\leqslant j}\{(v_i(x)v_j(y)+v_j(x)v_i(y)\}\frac{\partial}{\partial\tau_{ij}}\log\det(C\tau+D)dxdy, \tag{5.2.7}$$

where

$$\begin{pmatrix}\widehat{\beta}_1\\ \vdots\\ \widehat{\beta}_g\\ \widehat{\alpha}_1\\ \vdots\\ \widehat{\alpha}_g\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\widetilde{\beta}_1\\ \vdots\\ \widetilde{\beta}_g\\ \widetilde{\alpha}_1\\ \vdots\\ \widetilde{\alpha}_g\end{pmatrix} \tag{5.2.8}$$

and $\{\omega_1,\dots,\omega_N\}$ are normalized one forms of the family with $\omega_i = v_i(x)dx$. Now fix a family of symplectic bases

$$\{\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_g,\widetilde{\beta}_1,\dots,\widetilde{\beta}_g\}$$

and a basis $\{\omega_1,\dots,\omega_N\}$ of normalized one-forms of the family. Then as in section 1.4 we have a period mapping $\Pi\colon \mathbb{T}_g \to \mathfrak{S}_g$ and a group homomorphism $\Phi\colon \mathrm{Mod}_g \to Sp(g,\mathbb{Z})$ with the properties (1.4.7) and (1.4.8). Hence, for an element $\gamma\in\mathrm{Mod}_g$ we have a new symplectic basis $\{\widehat{\alpha}_1,\dots,\widehat{\alpha}_g,\widehat{\beta}_1,\dots,\widehat{\beta}_g\}$ by (5.2.8) and a new symmetric bidifferential $\gamma^*(\omega)$. Then, by (5.2.7) and (1.4.33) we have

$$S_{\gamma^*\omega}(z)dz^2 - S_\omega(z)dz^2 = -6\sum_{i\leqslant j}(v_i(z)v_j(z))\Pi^*\left(\frac{\partial}{\partial\tau_{ij}}\log\det(C_\gamma\tau+D_\gamma)\right)dz^2. \tag{5.2.9}$$

Hence, by Proposition 1.4.1 we have

$$S_{\gamma^*\omega}(z)dz^2 - S_\omega(z)dz^2 = 6\Pi^*(d\log\det(C_\gamma\tau+D_\gamma)).$$

For each element $\gamma\in\mathrm{Mod}_g$ put

$$f_\gamma := -\frac{c_v}{2}\Pi^*(d\log\det(C_\gamma\tau+D_\gamma)). \tag{5.2.10}$$

Then, $f_\gamma\in H^0(\mathbb{T}_g,\Omega^{1,cl}_{\mathbb{T}_g})$ and it is easy to show

$$f_{\gamma_1\gamma_2} - \gamma_2^* f_{\gamma_1} - f_{\gamma_2} = 0.$$

Therefore, $\{f_\gamma\}$ defines a one-cocycle of $H^1(\mathrm{Mod}_g, H^0(\mathbb{T}_g,\Omega^{1,cl}_{\mathbb{T}_g}))$. This is our class $\xi(\mathfrak{F})$. The exact sequence (5.2.3) over the Teichmüller space $\mathbb{T}_g$ is considered as an exact sequence of Mod_g-sheaves and we have a canonical homomorphism

$$H^1\left(\mathrm{Mod}_g, H^0(\mathbb{T}_g,\Omega^{1,cl}_{\mathbb{T}_g})\right)\xrightarrow{\widehat{c}} H^2(\mathrm{Mod}_g,\mathbb{C}^*).$$

The image $\widehat{c}(\xi(\mathfrak{F}))\in H^2(\mathrm{Mod}_g,\mathbb{C}^*)$ of $\xi(\mathfrak{F})$ is given by a two-cocycle

$$g_{\gamma_1\gamma_2} = \exp\Big(-\frac{c_v}{2}\{\log\det(C_{\gamma_3}\Pi(t)+D_{\gamma_3}) - \log\det(C_{\gamma_2}\Phi(\gamma_1)\cdot\Pi(t)+D_{\gamma_2}) - \log\det(C_{\gamma_1}\Pi(t)+D_{\gamma_1})\}\Big), \tag{5.2.11}$$

where we put $\gamma_3 = \gamma_1\cdot\gamma_2$. Since we have

$$\det(C_{\gamma_3}\Pi(t)+D_{\gamma_3}) = \det(C_{\gamma_2}\Phi(\gamma_1)\cdot\Pi(t)+D_{\gamma_2})\times\det(C_{\gamma_1}\Pi(t)+D_{\gamma_1}),$$

we conclude

$$g_{\gamma_1\gamma_2}\in\mu_{2(\ell+g^*)},$$

where $\mu_{2(\ell+g^*)}$ is the cyclic subgroup of $\mathbb{C}^*$ of order $2(\ell+g^*)$. Thus we obtain the following theorem.

Theorem 5.2.5. *Let* $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ *be a versal family of smooth curves of genus* g *over the the universal family over the Teichmüller space* $\mathbb{T}_g$. *Then the cohomology class* $\xi(\mathfrak{F})$ *associated to the sheaf of twisted differential operators* $\mathcal{D}_{\mathcal{B}}(c_v)$ *by Definition* 5.2.6 *is the pull back of the cohomology class*

$$\xi \in H^1\left(\mathrm{Mod}_g, H^0(\mathbb{T}_g, \Omega^{1,cl}_{\mathbb{T}_g})\right)$$

defined by (5.2.10). The characteristic class $\overline{c}(\xi(\mathfrak{F}))$ is the pull back of the cohomology class

$$\widehat{c}(\xi) \in H^2(\mathrm{Mod}_g, \mu_{2(\ell+g^*)})$$

defined by (5.2.11).

5.3. The Differential Equation Near the Boundary.

In this section we shall prove that the formal solutions constructed in (4.3.1) and (4.4.3) converge.

Let $\mathfrak{F}^{(0)} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N)$ be a versal family of N-pointed stable curves and $\mathfrak{F} = (\pi\colon \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ a family obtained by adding formal neighbourhoods to $\mathfrak{F}^{(0)}$. We use the same notation as that of section 4.3. Let D be the discriminant locus of the family $\mathfrak{F}^{(0)}$ and let $D = \bigcup_{i=1}^{k} D_i$ be the decomposition into irreducible components. In this section, for simplicity we assume $k = 1$. Put $E = D_i$. Denote by $\pi_E\colon \mathcal{C}_E \to E$ the restriction of $\mathcal{C}$ to E. Let $\widetilde{\pi}_E\colon \widetilde{\mathcal{C}}_E \to E$ be obtained by the simultaneous normalization of $\pi_E\colon \mathcal{C}_E \to E$ along the locus of double points over E and let $\sigma', \sigma''\colon E \to \widetilde{\mathcal{C}}_E$ be the cross-sections corresponding to the normalized double points over the locus E.

$$\begin{array}{ccccc}
\widetilde{\mathcal{C}}_E & \longrightarrow & \mathcal{C}_E & \hookleftarrow & \mathcal{C}^{(0)} \\
{\scriptstyle s_j,\sigma',\sigma''}\nwarrow\searrow{\scriptstyle \overline{\pi}_E} & & {\scriptstyle \pi_E}\downarrow\uparrow{\scriptstyle s_j,\sigma} & & \downarrow \\
 & & E & \hookleftarrow & \mathcal{B}.
\end{array}$$

As was shown in Proposition 4.4.1, The family

$$\widetilde{\mathfrak{F}}^{(0)}_E := (\widetilde{\pi}_E\colon \widetilde{\mathcal{C}}_E \to E; s_1, \ldots, s_N, \sigma', \sigma'')$$

is a local universal family of $(N+2)$-pointed (not necessarily connected) stable curves.

Lemma 5.3.1. *If we choose* $\mathcal{B}$ *sufficiently small, then there exist local coordinates* $(u_1, \ldots, u_M, z)$ *(respectively,* $(u_1, \ldots, u_M, w)$*) of a neighbourhood* X *(respectively,* Y*) of* $\sigma'(E)$ *(respectively,* $\sigma''(E)$*) in* $\widetilde{\mathcal{C}}_E$ *and a relative vector field*

$$\widetilde{\ell} \in H^0\left(\widetilde{\mathcal{C}}_E, \Theta_{\widetilde{\mathcal{C}}_E/E}\left(*\sum_{j=1}^{N} s_j(E)\right)\right),$$

which satisfy the following conditions.

(1) *The sections* σ' *and* σ'' *are given by the mappings*

$$\sigma'\colon (u_1, \ldots, u_M) \longmapsto (u_1, \ldots, u_M, 0) = (u_1, \ldots, u_M, z),$$
$$\sigma''\colon (u_1, \ldots, u_M) \longmapsto (u_1, \ldots, u_M, 0) = (u_1, \ldots, u_M, w).$$

(2) $\widetilde{\ell}|_X = \frac{1}{2} z \frac{\partial}{\partial z}, \quad \widetilde{\ell}|_Y = \frac{1}{2} w \frac{\partial}{\partial w}.$

Proof. Let $\nu\colon \widetilde{\mathcal{C}}_E \to \mathcal{C}_E$ be the simultaneous normalization along the double points over E. Let $(u_1,\dots,u_M,x)$ (respectively, $(u_1,\dots,u_M,y)$) be local coordinates of X (respectively, Y) satisfying the condition (1) in the above lemma. Since ν is an isomorphism (the identity mapping) on $\widetilde{\mathcal{C}}_E \setminus (\sigma'(E)\cup\sigma''(E)) = \mathcal{C}_E \setminus \sigma(E)$, by the proof of Lemma 1.3.3, by (1.3.4) in particular, we have the following exact sequence.

$$0 \longrightarrow \Theta_{\mathcal{C}_E/E} \longrightarrow \nu_*(\Theta_{\bar{\mathcal{C}}_E/E}(-\sigma'(E)-\sigma''(E))) \xrightarrow{\alpha} \mathcal{O}_E \longrightarrow 0,$$

where the $\mathcal{O}_E$-module homomorphism α is given by

$$\left(a(u,x)\frac{\partial}{\partial x}, b(u,y)\frac{\partial}{\partial y}\right) \longmapsto \frac{\partial a(u,0)}{\partial x} + \frac{\partial b(u,0)}{\partial y}.$$

Note that the stalk of $\nu_*(\Theta_{\bar{\mathcal{C}}_E/E}(-\sigma'(E)-\sigma''(E)))$ at a point $\sigma(u)$, $u\in E$ consists of a pair of local holomorphic vector fields $(a(u,x)\frac{\partial}{\partial x}, b(u,y)\frac{\partial}{\partial y})$ with $a(t,0)=0$, $b(y,0)=0$ and the definition of α is independent of the choice of local coordinates. The exact sequence induces an exact sequence

$$\begin{aligned} 0 \longrightarrow\ & H^0\left(\mathcal{C}_E, \Theta_{\mathcal{C}_E/E}(kS)\right) \\ & \longrightarrow H^0\left(\mathcal{C}_E, \nu_*\left(\Theta_{\bar{\mathcal{C}}_E/E}\left(\nu^*(kS)-\sigma'(E)-\sigma''(E)\right)\right)\right) \\ & \xrightarrow{\alpha} H^0(E,\mathcal{O}_E) \longrightarrow H^1\left(\mathcal{C}_E, \Theta_{\mathcal{C}_E/E}(kS)\right), \end{aligned}$$

for each integer k, where we put

$$S = \sum_{j=1}^{N} s_j(E).$$

If k is sufficiently large, we have

$$H^1\left(\mathcal{C}_E, \Theta_{\mathcal{C}_E/E}(kS)\right) = 0.$$

Hence, by the above exact sequence there exists a relative vector field

$$\begin{aligned} \widetilde{\ell} \in\ & H^0\left(\widetilde{\mathcal{C}}_E, \Theta_{\bar{\mathcal{C}}_E/E}\left(\nu^*(kS)-\sigma'(E)-\sigma''(E)\right)\right) \\ =\ & H^0\left(\mathcal{C}_E, \nu_*\left(\Theta_{\bar{\mathcal{C}}_E/E}\left(\nu^*(kS)-\sigma'(E)-\sigma''(E)\right)\right)\right), \end{aligned}$$

such that

$$\alpha(\widetilde{\ell}) \equiv 1.$$

In the local coordinates given above, $\widetilde{\ell}$ has the form

$$\begin{aligned} \widetilde{\ell} &= a(u,x)\frac{\partial}{\partial x} \quad \text{on } X, \\ \widetilde{\ell} &= b(u,y)\frac{\partial}{\partial y} \quad \text{on } Y \end{aligned}$$

with

$$\frac{\partial a(u,0)}{\partial x} + \frac{\partial b(u,0)}{\partial y} \equiv 1.$$

Adding an element coming from $H^0(\mathcal{C}_E, \Theta_{\mathcal{C}_E/E}(*S))$ if necessary, and choosing $\mathcal{B}$, X and Y smaller, we may assume that X and Y have Taylor expansions

$$\begin{aligned} a(u,x) &= \frac{1}{2}x + \sum_{n=2}^{\infty} a_n(u)x^n, \\ b(u,y) &= \frac{1}{2}y + \sum_{n=2}^{\infty} b_n(u)x^n. \end{aligned}$$

Now define $z = z(u,x)$ and $w = w(u,y)$ by

$$a(u,x)\frac{\partial z}{\partial x} = \frac{1}{2}z, \quad z(u,0) = 0,$$
$$b(u,y)\frac{\partial w}{\partial y} = \frac{1}{2}w, \quad w(u,0) = 0.$$

Then, by choosing X and Y smaller, $(u_1,\dots,u_M,z)$ and $(u_1,\dots,u_M,w)$ satisfy the above conditions (1) and (2). $\square$

Let $\widetilde{\mathfrak{F}}_E = (\widetilde{\pi}_E\colon \widetilde{\mathcal{C}}_E \to E; s_1, s_2, \dots, s_n, \sigma', \sigma''; \eta_1, \eta_2, \dots, \eta_N, z, w)$ be a family of $(N+2)$-pointed stable curves of genus g with formal coordinates obtained from $\widetilde{\mathfrak{F}}_E^{(0)}$.

Let $\ell_j(\xi_j)\frac{d}{d\xi_j}$ be the formal Laurent expansion of $\widetilde{\ell}$ with respect to the formal coordinate ξ_j. Thus we have $\ell_j(\xi_j) \in \mathcal{O}_E((\xi_j))$. Put

$$\underline{\vec{\ell}} = \left(\ell_1(\xi_1)\frac{d}{d\xi_1}, \dots, \ell_N(\xi_N)\frac{d}{d\xi_N}\right). \tag{5.3.2}$$

Next we reconstruct the family $\mathfrak{F}^{(0)}$ from the family $\widetilde{\mathfrak{F}}_E^{(0)}$. Using the notation of Lemma 5.3.1, we may assume that

$$X = \{\, P \in \widetilde{\mathcal{C}}_E \mid |z(P)| < 1 \,\},$$
$$Y = \{\, P \in \widetilde{\mathcal{C}}_E \mid |w(P)| < 1 \,\}.$$

For a positive number $\varepsilon < 1$ put

$$X_\varepsilon = \{\, P \in \widetilde{\mathcal{C}}_E \mid |z(P)| < \varepsilon \,\},$$
$$Y_\varepsilon = \{\, P \in \widetilde{\mathcal{C}}_E \mid |w(P)| < \varepsilon \,\}.$$

Fix positive numbers $\varepsilon_1 < \varepsilon_2 < 1$ and let $\{U_\alpha\}_{3 \leqslant \alpha \leqslant A}$ be a finite open covering of $\widetilde{\mathcal{C}}_E \setminus (X_{\varepsilon_2} \bigcup Y_{\varepsilon_2})$ such that

$$U_\alpha \cap X_{\varepsilon_1} = \varnothing, \quad U_\alpha \cap Y_{\varepsilon_1} = \varnothing$$

for any $\alpha = 3, \dots, A$.

Now put

$$D = \{\, \tau \in \mathbb{C} \mid |\tau| < 1 \,\}$$
$$S_0 = \{\, (x,y,\tau) \in \mathbb{C}^3 \mid xy = \tau, |x| < 1, |y| < 1, |\tau| < 1 \,\}$$
$$S = S_0 \times E$$
$$Z = \{\, (P,\tau) \in \widetilde{\mathcal{C}}_E \times D \mid P \in \widetilde{\mathcal{C}}_E \setminus (X \cup Y) \text{ or } P \in X \text{ and } |z(P)| > |\tau| \,\}$$
$$W = \{\, (P,\tau) \in \widetilde{\mathcal{C}}_E \times D \mid P \in \widetilde{\mathcal{C}}_E \setminus (X \cup Y) \text{ or } P \in Y \text{ and } |w(P)| > |\tau| \,\}.$$

On $Z \sqcup S \sqcup W$ we introduce an equivalence relation $\sim$ as follows

(1) A point $(P,\tau) \in Z \cap (X \times D)$ and a point $(x,y,\tau',u) \in S$ are equivalent if and only if

$$(x,y,\tau',u) = \left(z(P), \frac{\tau}{z(P)}, \tau, \widetilde{\pi}_E(P)\right).$$

(2) A point $(P,\tau) \in W \cap (Y \times D)$ and a point $(x,y,\tau',u) \in S$ are equivalent if and only if

$$(x,y,\tau',u) = \left(\frac{\tau}{w(P)}, w(P), \tau, \widetilde{\pi}_E(P)\right).$$

(3) A point $(P,\tau) \in Z$ and a point $(Q,\tau') \in W$ are equivalent if and only if

$$(P,\tau) = (Q,\tau').$$

Now put $\mathcal{C} = Z \sqcup S \sqcup W/\sim$. Then it is easy to show that $\mathcal{C}$ is a complex manifold and there is a natural holomorphic mapping $\pi\colon \mathcal{C} \to E \times D$. Moreover, since we can assume that $s_j(E)$'s are contained in $\widetilde{\mathcal{C}}_E \setminus (X \cup Y)$, we can define holomorphic sections s_j's by

$$\begin{array}{rccc} s_j\colon & E \times D & \longrightarrow & \mathcal{C}, \\ & (t,\tau) & \longmapsto & (s_j(t),\tau) \in Z. \end{array}$$

In the same way we can define the holomorphic coordinates η_j. It is easy to show that $(\pi\colon \mathcal{C} \to E \times D; s_1, \ldots, s_N)$ (respectively, $(\pi\colon \mathcal{C} \to E \times D; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$) is isomorphic to our original family $\mathfrak{F}^{(0)}$ (respectively, $\mathfrak{F}$). Hence, in the following we identify $\mathfrak{F}^{(0)}$ and $\mathfrak{F}$ with the families constructed above.

For each point $(u,\tau) \in E^{(1)} \times D$ put

$$\begin{gathered} \mathcal{C}_{(u,\tau)} = \pi^{-1}((u,\tau)), \\ U_\alpha(u,\tau) = U_\alpha \cap \mathcal{C}_{u,\tau}, \quad 3 \leqslant \alpha \leqslant A, \\ U_1(u,\tau) = S \cap Z \cap \mathcal{C}_{u,\tau}, \\ U_2(u,\tau) = S \cap W \cap \mathcal{C}_{u,\tau}. \end{gathered}$$

Then, for each $\tau \neq 0$, $\mathcal{U}(u,\tau) = \{U_\alpha(u,\tau)\}_{1 \leqslant \alpha \leqslant A}$ is an open covering of the curve $\mathcal{C}_{(u,\tau)}$.

Lemma 5.3.2. *For each point $(u,\tau) \in E \times D$ with $\tau \neq 0$, the image $\rho(\tau\frac{\partial}{\partial\tau})$ of the vector field $\tau\frac{\partial}{\partial\tau}$ by the Kodaira-Spencer mapping*

$$\rho\colon T_{(u,\tau)}(E \times D) \longrightarrow H^1(\mathcal{C}_{(u,\tau)}, \Theta_{\mathcal{C}_{(u,\tau)}})$$

is given by a Čech cohomology class $\{\theta_{\alpha\beta}(u,\tau)\} \in \check{H}^1(\mathcal{U}(u,\tau), \Theta_{\mathcal{C}_{(u,\tau)}})$ with respect to the covering $\mathcal{U}(u,\tau)$ given above, where

$$\begin{gathered} \theta_{12}(u,\tau) = z\frac{\partial}{\partial z}, \\ \theta_{21}(u,\tau) = -\theta_{12}(u,\tau), \\ \theta_{\alpha\beta}(u,\tau) = 0, \quad \text{if } (\alpha,\beta) \neq (1,2) \text{ or } (2,1). \end{gathered}$$

Proof. By the above equivalence relation, on $U_1(u,\tau) \cap U_2(u,\tau)$ we have

$$z = \frac{\tau}{w}.$$

If $U_\alpha(u,\tau) \cap U_\beta(u,\tau) \neq \varnothing$ and $(\alpha,\beta) \neq (1,2)$ nor $(2,1)$, then the relation between the local coordinates of $U_\alpha(u,\tau)$ and $U_\beta(u,\tau)$ does not depend on τ. Hence, by the definition of the Kodaira-Spencer mapping (see Remark 1.2.7.) we have

$$\begin{aligned} \rho\left(\tau\frac{\partial}{\partial\tau}\right)_{12} &= \frac{\tau}{w}\frac{\partial}{\partial\tau} = z\frac{\partial}{\partial z}, \\ \rho\left(\tau\frac{\partial}{\partial\tau}\right)_{21} &= w\frac{\partial}{\partial w} = -z\frac{\partial}{\partial z}, \\ \rho\left(\tau\frac{\partial}{\partial\tau}\right)_{\alpha\beta} &= 0, \quad \text{if } (\alpha,\beta) \neq (1,2) \text{ nor } (2,1). \end{aligned}$$

□

Let us consider an N-tuple of formal vector fields

$$\vec{\underline{\ell}} = \left(\ell_1(\xi_1)\frac{d}{d\xi_1}, \ldots, \ell_N(\xi_N)\frac{d}{d\xi_N}\right)$$

as defined in (5.3.2). Since we have $\ell_j(\xi_j)\frac{d}{d\xi_j} \in \mathcal{O}_E((\xi_j))$, we may regard $\vec{\underline{\ell}}$ as an N-tuple of formal vector fields on $\mathfrak{F}^{(0)}$, that is, $\ell_j(\xi_j)\frac{d}{d\xi_j} \in \mathcal{O}_{E\times D}((\xi_j))$.

Corollary 5.3.3. *On $\mathcal{B} = E \times D$ we have*

$$\theta(\underline{\vec{\ell}}) = \tau \frac{\partial}{\partial \tau},$$

where the mapping θ is defined in (4.2.6).

Proof. Since both sides of the above equality in the corollary defines holomorphic vector fields on $\mathcal{B}$, it is enough to prove the equality for $\tau \neq 0$.

Let us consider an exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-S) \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}((m-1)S) \longrightarrow \bigoplus_{j=1}^{N} \bigoplus^{m} \mathcal{O}_{\mathcal{B}} \xi_j^{-m+k} \frac{d}{d\xi_j} \longrightarrow 0,$$

for a sufficiently large positive integer m. $\underline{\vec{\ell}}$ defines an element $\underline{\vec{\ell}}'$ of the third term of the exact sequence. On the other hand, for each $(u, \tau) \in E \times D$, $\tau \neq 0$, the meromorphic vector field $\widetilde{\ell}$ on $\widetilde{\mathcal{C}}_E$ defines meromorphic vector fields $\widetilde{\ell}_{u,\tau}$ on $\mathcal{C}_{u,\tau} \setminus \{U_2(u,\tau) \setminus (U_1(u,\tau) \cap U_2(u,\tau))\}$ and $\widetilde{\ell}'_{u,\tau} = \frac{1}{2} w \frac{\partial}{\partial w}$ on $U_2(u,\tau)$ such that both vector fields have the same image $\underline{\vec{\ell}}'$ in the above exact sequence. Hence, the image of $\underline{\vec{\ell}}'$ by the mapping

$$\bigoplus_{j=1}^{N} \bigoplus^{m} \mathcal{O}_{\mathcal{B}} \xi_j^{-m+k} \frac{d}{d\xi_j} \longrightarrow R^1 \pi_* (\Theta_{\mathcal{C}/\mathcal{B}}(-S))$$

is given at a point (u, τ) by an element

$$\{\theta_{\alpha,\beta}(u,\tau)\} \in H^1(C_{(u,\tau)}, \Theta_{C_{(u,\tau)}}),$$

where on $U_1(u,\tau) \cap U_2(u,\tau)$ we have

$$\begin{aligned} \theta_{12}(u,\tau) &= \widetilde{\ell}_{u,\tau}|_{U_1(u,\tau)} - \widetilde{\ell}'_{u,\tau}|_{U_2(u,\tau)} \\ &= \frac{1}{2} z \frac{\partial}{\partial z} - \frac{1}{2} w \frac{\partial}{\partial w} \\ &= z \frac{\partial}{\partial z}, \\ \theta_{21}(u,\tau) &= -\theta_{12}(u,\tau) \end{aligned}$$

and on $U_\alpha(u,\tau) \cap U_\beta(u,\tau)$ with $(\alpha,\beta) \neq (1,2), (2,1)$ we have

$$\theta_{\alpha\beta}(u,\tau) = 0.$$

Thus $\underline{\vec{\ell}}$ defines the cohomology class given in Lemma 5.3.2. Hence we have the equality for $\tau \neq 0$. $\square$

Let $\tilde{\ell}$ be the meromorphic vector field with $\theta(\tilde{\ell})$ given in Lemma 5.3.1 and $\underline{\ell}_j$ be the formal Laurent expansion of $\tilde{\ell}$ at Q_j. Put $\underline{\vec{\ell}} = (\underline{\ell}_1, \ldots, \underline{\ell}_N)$ as in (5.3.2). Let $\langle \widetilde{\Psi}| \in \mathcal{H}^\dagger_{\vec{\lambda}}(E)[[\tau]]$ be an element defined in (4.4.3). Put

$$\langle \widehat{\Psi}| = \tau^{\Delta_\mu} \langle \widetilde{\Psi}|.$$

We shall show that $\langle \widehat{\Psi}|$ converges and is a multi-valued holomorphic section of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ near the boundary, by showing that $\langle \widehat{\Psi}|$ is a formal solution of a differential equation of Fuchsian type.

Theorem 5.3.4. *The formal power series $\langle\widehat{\Psi}|$ is a formal solution of the following differential equation of Fuchsian type*

$$\left(\tau\frac{d}{d\tau} - T[\underline{\vec{\ell}}] + a(\underline{\vec{\ell}})\right)\langle\widehat{\Psi}| = 0.$$

Moreover, $\langle\widehat{\Psi}|$ converges and defines a multi-valued holomorphic section of $\mathcal{V}^{\dagger}_{\lambda}(\mathfrak{F})$ near the boundary.

Proof. We may assume that $\mathcal{B}^{(\varnothing)}$ is small enough so that there exists a symmetric bidifferential $\omega \in H^0(\mathcal{C}^{(\varnothing)} \otimes_{\mathcal{B}^{(\varnothing)}} \mathcal{C}^{(\varnothing)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(\varnothing)}/\mathcal{B}^{(\varnothing)}}(2\Delta))$ with $\mathrm{Res}^2(\omega) = 1$. By (5.1.2) we have

$$\langle\widetilde{\Psi}|\widetilde{T}(u)|\Phi\rangle du^2 \in H^0\left(\mathcal{C}, \omega^{\otimes 2}_{\mathcal{C}/\mathcal{B}}(*S^{(1)})\right).$$

Let $\widetilde{\ell} = \ell(z)\frac{\partial}{\partial z}$ be the meromorphic vector field given in Lemma 5.3.1. Then, for $(u,\tau) \in E \times D$, $\tau \neq 0$,

$$\ell(z)\langle\widetilde{\Psi}|\widetilde{T}(z)|\Phi\rangle dz$$

is a meromorphic form on $C'_{(u,\tau)} = C_{(u,\tau)} \setminus \{\,(x,y,\tau) \in S_0 \mid |x| \leqslant \epsilon \text{ or } |y| \leqslant \epsilon\,\}$ for a sufficiently small positive number $\epsilon < 1$.

The boundary of $C'_{(u,\tau)}$ consists of two disjoint simple closed curves γ_+, γ_-. We choose the orientation of $\gamma_\pm$ in such a way that $C'_{u,\tau}$ lies in the right hand side of $\gamma_\pm$. Then by Theorem 3.4.3, (5.1.3) and (5.1.5) we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_+}\ell(z)\langle\widehat{\Psi}|\widetilde{T}(z)|\Phi\rangle dz + \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_-}\ell(w)\langle\widehat{\Psi}|\widetilde{T}(w)|\Phi\rangle dw\\
&\quad= \sum_{j=1}^{N}\mathrm{res}_{Q_j}(\ell(u)\langle\widehat{\Psi}|\widetilde{T}(u)|\Phi\rangle du)\\
&\quad= \sum_{j=1}^{N}\langle\widehat{\Psi}|\rho_j(\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)T(\xi_j)d\xi_j))|\Phi\rangle\\
&\qquad\qquad - \frac{c_v}{12}\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)S_\omega(\xi_j))\langle\widehat{\Psi}|\Phi\rangle\\
&\quad= \sum_{j=1}^{N}\langle\widehat{\Psi}|\rho_j(T[\ell_j])|\Phi\rangle - a(\underline{\vec{\ell}})\langle\widehat{\Psi}|\Phi\rangle.
\end{aligned}$$

On the other hand, on γ_+ we have $\ell(z)\frac{d}{dz} = \frac{1}{2}z\frac{d}{dz}$. Hence, by (5.1.2) we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_+}\ell(z)\langle\widehat{\Psi}|\widetilde{T}(z)|\Phi\rangle dz\\
&\quad= \frac{1}{4\pi\sqrt{-1}}\int_{\gamma_+}\left(\ell(z)\langle\widehat{\Psi}|T(z)|\Phi\rangle - \frac{c_v}{12}\ell(z)S_\omega(z)\langle\widehat{\Psi}|\Phi\rangle\right)dz\\
&\quad= \frac{1}{4\pi\sqrt{-1}}\int_{\gamma_+}\ell(z)\langle\widehat{\Psi}|T(z)|\Phi\rangle dz,
\end{aligned}$$

since $S_\omega(z)dz^2$ is holomorphic at $z = 0$. Hence, by (4.4.2), (4.4.3) and Theorem 3.4.3, item (1)

we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_+}\ell(z)\langle\widehat{\Psi}|\widetilde{T}(z)|\Phi\rangle dz\\
&\quad=\frac{1}{2}\sum_{d=0}^{\infty}\tau^{\Delta_\mu+d}\sum_{i=1}^{m_d}\int_{\gamma_+}z\langle\Psi_d|T(z)|v_i(d)\otimes v^i(d)\otimes\Phi\rangle dz\\
&\quad=\frac{1}{2}\sum_{d=0}^{\infty}\tau^{\Delta_\mu+d}\sum_{i=1}^{m_d}\langle\Psi_d|L_0(v_i(d))\otimes v^i(d)\otimes\Phi\rangle\\
&\quad=\frac{1}{2}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(\Delta_\mu+d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_-}\ell(w)\langle\widehat{\Psi}|\widetilde{T}(w)|\Phi\rangle dw\\
&\quad=\frac{1}{2}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(\Delta_{\mu^\dagger}+d)\tau^{\Delta_{\mu^\dagger}+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

Since we have $\Delta_\mu=\Delta_{\mu^\dagger}$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{N}\langle\widehat{\Psi}|\rho_j(T[\ell_j])|\Phi\rangle-a(\underline{\vec{\ell}})\langle\widehat{\Psi}|\Phi\rangle\\
&\qquad=\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(\Delta_\mu+d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\left\{\tau\frac{d}{d\tau}(\langle\widehat{\Psi}|)\right\}|\Phi\rangle\\
&\qquad=\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(\Delta_\mu+d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

Hence, $\langle\widehat{\Psi}|$ is a formal solution of the differential equation

$$\left(\tau\frac{d}{d\tau}\langle\widehat{\Psi}|\right)-(\langle\widehat{\Psi}|T[\underline{\vec{\ell}}])+a(\underline{\vec{\ell}})\langle\widehat{\Psi}|=0.$$

Finally we shall prove that $\langle\widehat{\Psi}|$ converges and defines a multi-valued section of $\mathcal{V}^\dagger_{\underline{\lambda}}(\mathfrak{F})$ near the boundary. By Theorem 4.4.2 $\mathcal{V}^\dagger_{\underline{\lambda}}(\mathfrak{F})$ is locally free. Fix a local trivialization

$$p\colon\mathcal{V}^\dagger_{\underline{\lambda}}(\mathfrak{F})\simeq\mathcal{O}_{\mathcal{B}}^{\oplus n}.$$

Then for the differential operator $D:=\tau\frac{d}{d\tau}-T[\underline{\vec{\ell}}]+a([\underline{\vec{\ell}}])$ acting on $\mathcal{V}^\dagger_{\underline{\lambda}}(\mathfrak{F})$ from the left, the operator $P:=p\circ D\circ p^{-1}$ acting on $\mathcal{O}_{\mathcal{B}}^{\oplus n}$ is a system of first order differential equations of Fuchsian type. On the other hand, by Lemma 4.4.6 the local trivialization p induces a local trivialization of the sheaf $\widehat{\mathcal{V}}^\dagger_{\underline{\lambda}/E}$

$$\widehat{p}\colon\widehat{\mathcal{V}}^\dagger_{\underline{\lambda}/E}\simeq\widehat{\mathcal{O}}_{\mathcal{B}/E}^{\oplus n}.$$

Thus the formal solution $\langle\widehat{\Psi}|$ may be considered as the n-tuple of formal power series which is the solution of the system of the differential equations P of Fuchsian type. By the theory of differential equations of Fuchsian type, the formal solution always converges. Thus, $\langle\widehat{\Psi}|$ determines a multi-valued section of $\mathcal{V}^\dagger_{\underline{\lambda}}(\mathfrak{F})$ near the boundary. □

For the theory of differential equations of Fuchsian type we refer the reader to [6, chapter 4].

5.4. The Differential Equation for $g = 0$.

In this section we shall study the differential equations for the spaces of vacua in the case of $g = 0$. Fix a positive integer $N \geqslant 3$ and put

$$B_N = \mathbb{C}^N - \bigcup_{i<j} \Delta_{ij},$$

where

$$\Delta_{ij} = \{\, (z_1, \ldots, z_N) \in \mathbb{C}^N \mid z_i = z_j \,\}.$$

We consider $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$ and in the following we fix a coordinate u on $\mathbb{C}$. We let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ be a family of N-pointed Riemann spheres where

$$\begin{gathered}
\mathcal{B} = B_N, \\
\mathcal{C} = B_N \times \mathbb{P}^1, \\
\pi : \mathcal{C} \longrightarrow \mathcal{B} \quad \text{the projection to the second factor,} \\
s_j : \mathcal{B} \longrightarrow \mathcal{B} \times \mathbb{P}^1, \\
(z_1, \ldots, z_N) \longmapsto (z_1, \ldots, z_N, (z_j : 1)),
\end{gathered}$$

and

$$\eta_j : \widehat{\mathcal{O}}_{\mathcal{C}/s_j(\mathcal{B})} \longrightarrow \mathcal{O}_{\mathcal{B}}[[\xi]]$$

is given by

$$\eta_j^{-1}(\xi) = \xi_j = u - z_j.$$

Since the coordinates η_j is determined by s_j, (note that we fix the coordinate u of $\mathbb{C}$) the family $\mathfrak{F}$ may be regarded as a family of N-pointed smooth curves of genus 0. Note that our family is not versal, since our family has a N-dimensional parameter space. The parameter space of a versal family of N-pointed smooth curve of genus 0 is of dimension $N - 3$. In the following we define the integrable connection on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ over B_N. The differential equations of the flat sections is nothing but the Knizhnik-Zamolodchikov equations.

Put

$$\begin{gathered}
\Lambda = \{1, 2, \ldots, N, \infty\}, \\
\zeta_\lambda = u - z_\lambda, \qquad \text{for } \lambda = 1, 2, \ldots, N, \\
\zeta_\infty = 1/u,
\end{gathered}$$

and introduce an open covering $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ defined by

$$\mathcal{U}_\lambda = \{\, (z_1, \ldots, z_N; \zeta_\lambda) \mid (z_1, \ldots, z_N) \in B_N, |\zeta_\lambda| < \epsilon \,\},$$

for a suitable positive number ϵ.

Lemma 5.4.1. *The Kodaira-Spencer mapping*

$$\rho : \Theta_{\mathcal{B}} \longrightarrow R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}\Big(-\sum_{j=1}^{N} s_j(\mathcal{B})\Big)$$

of the family $\mathfrak{F}$ is given by

$$\theta = \sum a_j(z) \frac{\partial}{\partial z_j} \longmapsto [\{\rho_{\lambda\mu}\}]$$

where on $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \varnothing$, $\lambda \neq \mu$, $\rho_{\lambda\mu}$ *is given by*

$$\rho_{\lambda\mu} = \begin{cases} (a_\mu(z) - a_\lambda(z))\frac{\partial}{\partial\zeta_\lambda} & \text{if } 1 \leqslant \lambda, \mu \leqslant N, \\ -a_\lambda(z)\frac{\partial}{\partial\zeta_\lambda}, & \text{if } \mu = \infty. \end{cases}$$

Moreover, $\operatorname{Ker}\rho$ *is an* $\mathcal{O}_\mathcal{B}$*-free module of rank 3 spanned by the global sections*

$$\sum_{j=1}^{N} z_j^k \frac{\partial}{\partial z_j}, \quad k = 0, 1, 2.$$

Proof. For $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \varnothing$, $\lambda \neq \mu$, and $1 \leqslant \lambda, \mu \leqslant N$ we have

$$\zeta_\lambda = \zeta_\mu + z_\mu - z_\lambda.$$

Hence, by section 1.2, we have

$$\rho_{\lambda\mu} = (a_\mu(z) - a_\lambda(z))\frac{\partial}{\partial\zeta_\lambda}.$$

On the other hand, if $\mu = \infty$ and $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \varnothing$,

$$\zeta_\lambda = 1/\zeta_\infty - z_\lambda.$$

Thus

$$\rho_{\lambda\infty} = -a_\lambda(z)\frac{\partial}{\partial\zeta_\lambda}.$$

This proves the first part of the lemma.

Next assume that $[\{\rho_{\lambda\mu}\}]$ is zero. Then, there exist

$$\theta_\lambda \in \Gamma\Big(\mathcal{U}_\lambda, \pi_*\Theta_{\mathcal{C}/\mathcal{B}}\Big(-\sum s_j(\mathcal{B})\Big)\Big),$$

such that

$$\begin{gathered} \theta_\lambda = \vartheta_\lambda(z, \zeta_\lambda)\frac{\partial}{\partial\zeta_\lambda}, \\ \vartheta_\lambda(z, 0) = 0, \\ \rho_{\lambda\mu} = \theta_\mu - \theta_\lambda. \end{gathered} \tag{5.4.1}$$

Then, if $1 \leqslant \lambda, \mu \leqslant N$, we have

$$a_\mu(z)\frac{\partial}{\partial\zeta_\mu} - \theta_\lambda = a_\lambda(z) - \frac{\partial}{\partial\zeta_\lambda}$$

and if $\mu = \infty$,

$$\theta_\infty = a_\lambda(z)\frac{\partial}{\partial\zeta_\lambda} - \theta_\lambda.$$

Hence,

$$\left\{a_\lambda(z)\frac{\partial}{\partial\zeta_\lambda} - \theta_\lambda\right\}$$

is a global vector field on $\mathcal{C}$, and has the form

$$(A(z) + B(z)u + C(z)u^2)\frac{\partial}{\partial u}.$$

Therefore we have

$$\begin{aligned}\vartheta_\lambda(z,\zeta_\lambda) &= (a_\lambda(z) - \{A(z) + B(z)(\zeta_\lambda + z_\lambda) + C(z)(\zeta_\lambda + z_\lambda)^2\} \\ &= a_\lambda(z) - \{A(z) + B(z)z_\lambda + C(z)z_\lambda^2\} + O(\zeta_\lambda).\end{aligned}$$

Hence, by (5.4.1) and by putting $\zeta_\lambda = 0$, we conclude

$$a_\lambda(z) = \{A(z) + B(z)z_\lambda + C(z)z_\lambda^2\}$$

□

The exact sequence

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}}\Big(-\sum s_j(\mathcal{B})\Big) \longrightarrow \Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}\frac{d}{d\xi_j} \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \pi_*\Theta_{\mathcal{C}/\mathcal{B}} \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}\frac{d}{d\xi_j} \xrightarrow{\vartheta} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}\Big(-\sum s_j(\mathcal{B})\Big) \longrightarrow 0.$$

Note that $\pi_*\Theta_{\mathcal{C}/\mathcal{B}}$ is a free $\mathcal{O}_{\mathcal{B}}$-module of rank three and we can choose

$$u^k\frac{\partial}{\partial u}, \quad k = 0, 1, 2,$$

as a basis of the $\mathcal{O}_{\mathcal{B}}$-module. For an element $\vec{\ell} \in \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}\frac{d}{d\xi_j}$, the image $\vartheta(\vec{\ell})$ is given by a one cocycle $\{\vartheta_{\lambda\mu}\}$, where

$$\vartheta_{\lambda\mu} = \begin{cases} \underline{\ell}_\mu - \underline{\ell}_\lambda & \text{if } 1 \leqslant \lambda, \mu \leqslant N \\ -\underline{\ell}_\lambda & \text{if } \mu = \infty. \end{cases}$$

By Lemma 5.4.1 we have the following lemma.

Lemma 5.4.2. *There is a natural $\mathcal{O}_{\mathcal{B}}$-module isomorphism*

$$\begin{aligned}\theta\colon \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}\frac{d}{d\xi_j} &\longrightarrow \Theta_{\mathcal{B}}, \\ \sum_{j=1}^{N} a_j(z)\frac{d}{d\xi_j} &\longmapsto \sum_{j=1}^{N} a_j(z)\frac{\partial}{\partial z_j},\end{aligned}$$

such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_*\Theta_{\mathcal{C}/\mathcal{B}} & \longrightarrow & \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}\frac{d}{d\xi_j} & \xrightarrow{\vartheta} & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-\sum s_j(\mathcal{B})) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\theta & & \| & & \\ 0 & \longrightarrow & \operatorname{Ker}\rho & \longrightarrow & \Theta_{\mathcal{B}} & \xrightarrow{\rho} & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-\sum s_j(\mathcal{B})) & \longrightarrow & 0, \end{array}$$

where the isomorphism θ on $\pi_*\Theta_{\mathcal{C}/\mathcal{B}}$ has the form

$$\theta\left(u^k\frac{\partial}{\partial u}\right) = \sum_{j=1}^{N} z_j^k\frac{\partial}{\partial z_j}.$$

Definition 5.4.3. For an element $\vec{\ell} = (\underline{\ell}_1, \dots, \underline{\ell}_N) \in \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}} \frac{d}{d\xi_j}$ we define the action of $D(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}(\mathfrak{F})$ by

$$D(\vec{\ell})(F \otimes |\Phi\rangle) = \theta(\vec{\ell})(F) \otimes |\Phi\rangle + F \otimes \Big(\sum_{j=1}^N \rho_j(T[\underline{\ell}_j])|\Phi\rangle\Big). \tag{5.4.2}$$

Proposition 5.4.4. *For each element $\vec{\ell} \in \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}} \frac{d}{d\xi_j}$ we have*

$$D(\vec{\ell})(\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathfrak{F})) \subset \widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathfrak{F}).$$

Proof. The sheaf $\pi_*\mathcal{O}_{\mathcal{B}}\big(*\sum s_j(\mathcal{B})\big)$ is spanned by

$$\frac{1}{(u - z_i)^m}, \quad i = 1, 2, \dots, N, \quad m = 0, 1, 2, \dots.$$

By the proof of Proposition 4.2.3, in particular by (4.2.11), it is enough to show that for $\vec{\ell} = (\underline{\ell}_1, \dots, \underline{\ell}_N) \in \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}} \frac{d}{d\xi_j}$ and for $h \in \pi_*\mathcal{O}_{\mathcal{C}}(*\sum s_j(\mathcal{B}))$ we have

$$(\theta(\vec{\ell}) + \underline{\ell}_j)(h) \in \pi_*\mathcal{O}_{\mathcal{C}}\Big(*\sum s_j(\mathcal{B})\Big).$$

For that purpose, it suffices to prove that for $A(z) \in \mathcal{O}_{\mathcal{B}}$,

$$(\theta(\vec{\ell}) + \underline{\ell}_j)\left(\frac{A(z)}{(\xi_j + -z_j - z_i)^m}\right) \in \pi_*\mathcal{O}_{\mathcal{C}}\Big(*\sum s_j(\mathcal{B})\Big).$$

Put

$$\underline{\ell}_j = c_j(z)\frac{d}{d\xi_j}, \quad j = 1, 2, \dots, N.$$

Then, by Lemma 5.4.2,

$$\theta(\vec{\ell})\left(t_j\left(\frac{A(z)}{(u - z_i)^m}\right)\right) = \frac{-\sum_{j=1}^N c_j(z)\frac{\partial A(z)}{\partial z_i}}{(u - z_i)^m} + \frac{(mc_j(z) - c_i(z))A(z)}{(u - z_i)^{m+1}},$$
$$\underline{\ell}_j\left(\frac{A(z)}{(u - z_i)^m}\right) = \frac{-mc_j(z)A(z)}{(u - z_i)^{m+1}}.$$

Thus we have

$$(\theta(\vec{\ell}) + \underline{\ell}_j)\left(t_j\left(\frac{A(z)}{(u - z_j)^m}\right)\right) = \frac{-\sum_{j=1}^N c_j(z)\frac{\partial A(z)}{\partial z_j}}{(u - z_j)^m} - \frac{mc_i(z)A(z)}{(u - z_i)^{m+1}}.$$

The right hand side is independent of j and defines an element of $\pi_*\mathcal{O}_{\mathcal{C}}(*\sum s_j(\mathcal{B}))$. □

Corollary 5.4.5. *For each element $\vec{\ell} \in \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}} \frac{d}{d\xi_j}$, $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ as a twisted first order differential operator.*

Lemma 5.4.6. *The vector space*

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \omega_{\mathbb{P}^1 \times \mathbb{P}^1}(2\Delta))$$

is one-dimensional and spanned by the element

$$\omega = \frac{dwdz}{(w - z)^2}. \tag{5.4.3}$$

Hence in particular

$$S_\omega \equiv 0.$$

By this lemma in our situation we can define an integrable connection on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$. We shall explicitly write down the connection. For any $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$, $v \in V_{\vec{\lambda}}$ and $\vec{\ell} \in \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}} \frac{d}{d\xi_j}$, by (4.2.6) and (4.2.8) we have

$$\langle\Psi|\{D(\vec{\ell})|v\rangle\} = \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|v\rangle d\xi_j),$$

$$\{D(\vec{\ell})(\langle\Psi|)\}|v\rangle = -\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|v\rangle d\xi_j) + \theta(\vec{\ell})(\langle\Psi|v\rangle).$$

Note that by the argument of section 3.5, $\langle\Psi|D(\vec{\ell}) = 0$, if and only if $\{\langle\Psi|D(\vec{\ell})\}|v\rangle = 0$ for all $v \in V_{\vec{\lambda}}$. Thus by the above equalities we conclude that $\langle\Psi|D(\vec{\ell}) = 0$, if and only if the following equality holds.

$$\theta(\vec{\ell})\langle\Psi|v\rangle = \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|v\rangle d\xi_j).$$

By Lemma 5.4.2 and (3.5.14) the differential equations defining flat sections are the following

$$\begin{aligned}\frac{\partial}{\partial z_j}\langle\Psi|v\rangle &= \sum_{j+1}^N \operatorname*{Res}_{\xi_j=0}(\langle\Psi|T(\xi_j)|v\rangle d\xi_j) \\ &= \frac{1}{2(g^*+\ell)}\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}\Big(\sum_{i=1}^N\sum_{k=1}^N\Big(\frac{\langle\Psi|\Omega_{ik}|v\rangle}{(\xi_j+z_j-z_i)(\xi_j+z_j-z_k)}\Big)d\xi_j\Big) \\ &= \frac{1}{(g^*+\ell)}\sum_{k=1,k\neq j}^N \frac{\langle\Psi|\Omega_{jk}|v\rangle}{z_j-z_k}.\end{aligned}$$

These equations are the Knizhnik-Zamolodchikov equations [24].

References

1. M. Artin, *Lectures on deformation of singularities*, Tata Institute Lecture Notes **54** (1976).
2. A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), 385–419.
3. A. A. Beilinson and V. V. Shechtman, *Determinant bundles and Virasoro algebra*, Comm. Math. Phys. **118** (1988), 651–701.
4. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984), 333–380.
5. D. Bernard and G. Felder, *Fock representations and* BRST *cohomology in* $SL(2)$ *current algebra*, Comm. Math. Phys. **127** (1990), 145–168.
6. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955.
7. C. De Concini, V. Kac and Kazhdan, *Boson-fermion correspondence over* $\mathbb{Z}$, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys., vol. **7**, World Scientific, 1989, pp. 603–622.
8. P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Pub. Math. IHES **36** (1969), 75–109.
9. R. Dijkgraaf and E. Verlinde, *Modular invariance and the fusion algebra*, Nucl. Phys. B(Proc. Suppl.) **5B** (1988), 87–97.
10. T. Eguchi and H. Ooguri, *Conformal and current algebras on a general Riemann surface*, Nucl. Phys. **B282** (1987), 308–328.
11. J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Math., vol. 352, Springer-Verlag, 1973.
12. G. Felder, BRST *approach to minimal models*, Nucl. Phys. **B317** (1989), 215–236.

13. D. Friedan and S. Shenker, *The analytic geometry of two-dimensional conformal field theory*, Nucl. Phys. **B281** (1987), 509–545.
14. O. Gabber, *The integrability of the characteristic variety*, Amer. J. Math. **103** (1981), 445–468.
15. H. Garland, *The arithmetic theory of loop algebra*, J. Algebra **53** (1978), 480–551.
16. L. Gerritzen, F. Herrich and M. van der Put, *Stable n-pointed trees of projective lines*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), 131–163.
17. R. Hartshorn, *Algebraic Geometry*, Springer-Verlag, 1977.
18. N. J. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys. **131** (1990), 347–380.
19. V. Kac, *Infinite dimensional Lie algebras, third edition*, Cambridge Univ. Press, 1990.
20. T. Katsura, Y. Shimizu and K. Ueno, *New bosonization and conformal field theory over* $\mathbb{Z}$, Comm. Math. Phys. **121** (1988), 603–622.
21. ———, *Formal groups and conformal field theory over* $\mathbb{Z}$, Advanced Studies in Pure Mathematics **19** (1988), Kinokuniya Shoten and Academic Press, 347–366.
22. ———, *Complex cobordism ring and conformal field theory over* $\mathbb{Z}$, Math. Ann. **291** (1991), 551–571.
23. N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, *Geometric realization of conformal field theory on Riemann surfaces*, Comm. Math. Phys. **116** (1988), 247–308.
24. V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B247** (1984), 83–103.
25. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves I*, Math. Scand. **89** (1976), 19–55.
26. K. Kodaira, *Complex manifolds and Deformation of Complex Structures*, Springer-Verlag, 1985.
27. T. Kohno, *Three-manifold invariants derived from conformal field theory and projective representations of modular groups*, Intern. J. Modern Phys. **6** (1992), 1795–1805.
28. S. Kumar, M. S. Narasimhan and A. Ramanathan, *Infinite Grassmannian and moduli space of G-bundles*, preprint (1993).
29. G. Kuroki, *Fock space representations of affine Lie algebras and integral representations in the Wess-Zumino-Witten models*, Comm. Math. Phys. **142** (1991), 511–542.
30. Y. Laszlo and C. Sorger, *The line bundles on the stack of parabolic G-bundles over curves and their sections*, preprint (1995).
31. G. Moore and N. Seiberg, *Polynomial equations for rational conformal field theories*, Phys. Lett. **B212** (1988), 451–460.
32. ———, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254.
33. G. Segal, *The definition of conformal field theory*, preprint.
34. T. Suzuku, *Finite-dimensionality of the space of conformal blocks*, preprint (1994).
35. T. Terada, *Quelques propiétés géométriques du domaine de* F_1 *et le groupe de tresses colorées*,, Publ. Res. Inst. Math. Sci. **17** (1981), 95–111.
37. S. Tsuboi, *Deformations of locally stable holomorphic maps and locally trivial displacement of analytic subvarieties with ordinary singularities*, Science Report of Kagoshima Univ. **35** (1986), 9–90.
36. A. Tsuchiya and Y. Kanie, *Vertex Operators in Conformal Field Theory on* $\mathbb{P}^1$ *and Monodromy Representations of Braid Group*, Advanced Studies in Pure Mathematics **16** (1988), Kinokuniya Shoten and Academic Press, 297–326.
38. A. Tsuchiya, K. Ueno and Y. Yamada, *Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries*, Advanced Studies in Pure Mathematics **19** (1989), Kinokuniya Shoten and Academic Press, 459–566.
39. K. Ueno, *On conformal field theory*, London Math. Soc. Lecture Note (1995), Cambridge Univ. Press, 283–345.
40. E. Verlinde, *Fusion rules and modular transformations in 2-D conformal field theory*, Nucl. Phys. **B300** [**FS22**] (1988), 360–376.
41. E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

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